Appendix B

Poincaré group

Poincaré invariance is the fundamental symmetry in particle physics. A relativistic quantum field theory must have a Poincaré-invariant action. This means that its fields must transform under representations of the Poincaré group and Poincaré invariance must be implemented unitarily on the state space. Here we will collect some properties of the Lorentz and Poincaré groups together with their representation theory.

B.1 Lorentz and Poincaré group

Lorentz group. We work in Minkowski space with the metric tensor $g = (g_{\mu\nu}) = \text{diag}\,(1,-1,-1,-1)$, where the scalar product is given by

$$x \cdot y := x^T g\, y = x^0 y^0 - \boldsymbol{x} \cdot \boldsymbol{y} = g_{\mu\nu}\, x^\mu y^\nu = x_\mu\, y^\mu. \tag{B.1}$$

Instead of carrying around explicit instances of g, it is more convenient to use the index notation where upper and lower indices are summed over. Lorentz transformations are those transformations $x' = \Lambda x$ that leave the scalar product invariant:

$$(\Lambda x) \cdot (\Lambda y) = x \cdot y \quad \Rightarrow \quad x^T \Lambda^T g\, \Lambda\, y = x^T g\, y \quad \Rightarrow \quad \Lambda^T g\, \Lambda = g\,. \tag{B.2}$$

Written in components, this condition takes the form

$$g_{\alpha\beta} = g_{\mu\nu}\, \Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\,. \tag{B.3}$$

Since the metric tensor is symmetric, this gives 10 constraints; the Lorentz transformation Λ is a 4×4 matrix, so it depends on $16 - 10 = 6$ independent parameters. If we write an infinitesimal transformation as $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \varepsilon^\alpha{}_\beta + \dots$, then it follows from Eq. (B.3) that $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ must be totally antisymmetric.

The transformations of a space with coordinates $\{y_1 \dots y_n, x_1 \dots x_m\}$ that leave the quadratic form $(y_1^2 + \dots + y_n^2) - (x_1^2 + \dots + x_m^2)$ invariant constitute the orthogonal group $O(m,n)$, so the Lorentz group is $O(3,1)$. The group axioms are satisfied; there

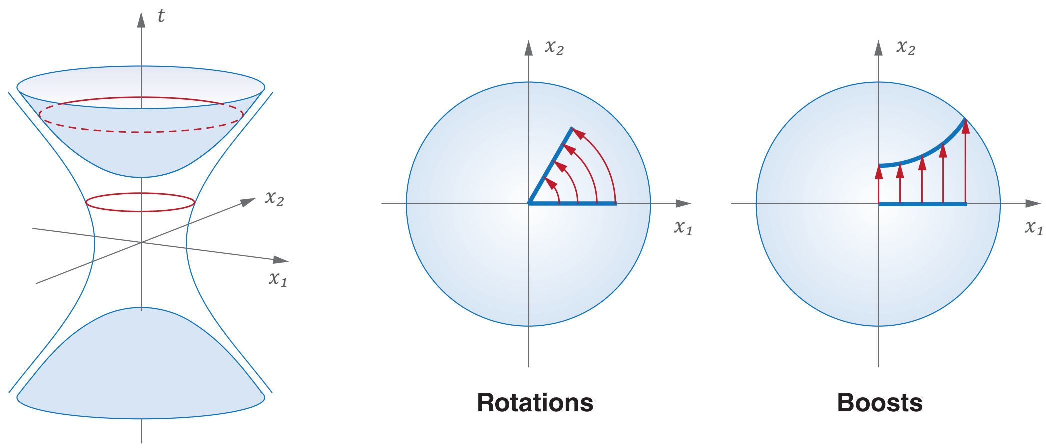

FIGURE B.1: Invariant hyperboloids for the Lorentz group. Rotations go around circles and boosts in fixed directions $\boldsymbol{n}$ along the surface.

is a unit element ($\Lambda = 1$), and each Λ has an inverse element because it is invertible: $\Lambda^T g\,\Lambda = g \Rightarrow (\det\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$. Eq. (B.3) also entails

$$g_{\mu\nu}\,\Lambda^\mu{}_0\,\Lambda^\nu{}_0 = (\Lambda^0_0)^2 - \sum_k (\Lambda^k{}_0)^2 = 1 \quad \Rightarrow \quad (\Lambda^0_0)^2 \geq 1\,. \tag{B.4}$$

Depending of the signs of $\det\Lambda$ and Λ^0_0, the Lorentz group has four disconnected components. The subgroup with $\det\Lambda = 1$ and $\Lambda^0_0 \geq 1$ is called the proper orthochronous Lorentz group $SO(3,1)^\uparrow$; it contains the identity matrix and preserves the direction of time and parity. The other three branches can be constructed from a given $\Lambda \in SO(3,1)^\uparrow$ combined with a space and/or time reflection:

- $SO(3,1)^\uparrow \times$ spatial reflections: $\Lambda^0_0 \geq 1$, $\det\Lambda = -1$
- $SO(3,1)^\uparrow \times$ time reversal: $\Lambda^0_0 \leq -1$, $\det\Lambda = -1$
- $SO(3,1)^\uparrow \times$ spacetime reflection: $\Lambda^0_0 \leq -1$, $\det\Lambda = 1$

Lorentz transformations preserve the norm $x^2 = x \cdot x$ in Minkowski space, which is positive for timelike four-vectors, negative for spacelike vectors, or zero for lightlike vectors. Therefore, they are transformations along the hypersurfaces of constant norm (Fig. B.1). For a four-momentum with positive norm $p^2 = m^2$ these are the forward and backward mass shells. For vanishing norm the hypersurface becomes the light cone, and for negative norm the hyperboloid lies outside of the light cone.

Each $\Lambda \in SO(3,1)^\uparrow$ can be reconstructed from a Lorentz boost with velocity $\beta = \frac{v}{c}$ in direction $\boldsymbol{n}$ (with $|\beta| < 1$) together with a spatial rotation $R(\boldsymbol{\alpha}) \in SO(3)$:

$$\Lambda = \underbrace{\left(\begin{array}{c|c} \gamma & \gamma\,\beta\,\boldsymbol{n}^T \\ \hline \gamma\,\beta\,\boldsymbol{n} & \mathbb{1} + (\gamma-1)\,\boldsymbol{n}\,\boldsymbol{n}^T \end{array}\right)}_{\mathsf{L}(\boldsymbol{\beta})} \underbrace{\left(\begin{array}{c|c} 1 & \boldsymbol{0}^T \\ \hline \boldsymbol{0} & R(\boldsymbol{\alpha}) \end{array}\right)}_{\mathsf{R}(\boldsymbol{\alpha})}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,. \tag{B.5}$$

In the nonrelativistic limit $|\beta| \ll 1 \Rightarrow \gamma \approx 1$ this recovers the Galilei transformation.

The six group parameters can therefore be chosen as the three components of the velocity $\beta\boldsymbol{n}$ and the three rotation angles $\boldsymbol{\alpha}$. One can show that interchanging the order in Eq. (B.5) yields

$$\Lambda = \mathsf{L}(\boldsymbol{\beta})\,\mathsf{R}(\boldsymbol{\alpha}) = \mathsf{R}(\boldsymbol{\alpha})\,\mathsf{L}\big(\mathsf{R}(\boldsymbol{\alpha})^{-1}\,\boldsymbol{\beta}\big)\,. \tag{B.6}$$

The rotation group $SO(3)$ forms a subgroup of the Lorentz group (two consecutive rotations form another one) whereas boosts do not: the product of two boosts generally also involves a rotation as in Eq. (B.6). There are two properties that will become important later in the context of representations: the Lorentz group is not compact because it contains boosts (hence all unitary representations are infinite-dimensional); and it is not simply connected because it contains rotations (so we need to study the representations of its universal covering group $SL(2,\mathbb{C})$).

Poincaré group. Actually, the fact that the Lorentz group leaves the norm x^2 of a vector invariant is not enough because on physical grounds we need the line element $(dx)^2 = g_{\mu\nu}\,dx^\mu dx^\nu = c^2(dt)^2 - (d\boldsymbol{x})^2$ to be invariant. This guarantees that the speed of light is the same in every inertial frame, and it allows us to add constant translations to the Lorentz transformation:

$$x' = T(\Lambda,a)\,x = \Lambda x + a. \tag{B.7}$$

The resulting 10-parameter group which contains translations, rotations and boosts is the Poincaré group or inhomogeneous Lorentz group. We can check again that the group axioms are satisfied: two consecutive Poincaré transformations form another one,

$$T(\Lambda',a')\,T(\Lambda,a) = T(\Lambda'\Lambda, a' + \Lambda' a)\,, \tag{B.8}$$

the transformation is associative: $(T\,T')\,T'' = T\,(T'\,T'')$, the unit element is $T(1,0)$, and by equating Eq. (B.8) with $T(1,0)$ we can read off the inverse element:

$$T^{-1}(\Lambda,a) = T(\Lambda^{-1}, -\Lambda^{-1}a)\,. \tag{B.9}$$

In analogy to above, the component which contains the identity $T(1,0)$ is called $ISO(3,1)^\uparrow$, where I stands for *inhomogeneous*. This is the fundamental symmetry group of physics that transforms inertial frames into one another.

Poincaré algebra. Consider now the representations $U(\Lambda,a)$ of the Poincaré group on some vector space. They inherit the transformation properties from Eqs. (B.8–B.9), and we use the symbol U although they are not necessarily unitary. The Poincaré group $ISO(3,1)^\uparrow$ is a Lie group and therefore its elements can be written as

$$U(\Lambda,a) = e^{\frac{i}{2}\varepsilon_{\mu\nu}\,M^{\mu\nu}}\,e^{ia_\mu P^\mu} = 1 + \tfrac{i}{2}\,\varepsilon_{\mu\nu}\,M^{\mu\nu} + ia_\mu P^\mu + \dots\,, \tag{B.10}$$

where the explicit forms of $U(\Lambda,a)$ and the generators $M^{\mu\nu}$ and P^μ depend on the representation. Since $\varepsilon_{\mu\nu}$ is totally antisymmetric, $M^{\mu\nu}$ can also be chosen antisymmetric. It contains the six generators of the Lorentz group, whereas the momentum operator

P^μ is the generator of spacetime translations. $M^{\mu\nu}$ and P^μ form a Lie algebra whose commutator relations can be derived from

$$U(\Lambda,a)\,U(\Lambda',a')\,U^{-1}(\Lambda,a) = U(\Lambda\Lambda'\Lambda^{-1},\, a+\Lambda a' - \Lambda\Lambda'\Lambda^{-1}a)\,, \tag{B.11}$$

which follows from the composition rules (B.8) and (B.9). Inserting infinitesimal transformations (B.10) for each $U(\Lambda = 1+\varepsilon, a)$, with $U^{-1}(\Lambda,a) = U(1-\varepsilon,-a)$, keeping only linear terms in all group parameters ε, ε', a and a', and comparing coefficients of the terms $\sim \varepsilon\varepsilon'$, $a\varepsilon'$, $\varepsilon a'$ and aa' leads to the identities

$$\begin{aligned} i\big[M^{\mu\nu}, M^{\rho\sigma}\big] &= g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\,, &\text{(B.12)}\\ i\big[P^\mu, M^{\rho\sigma}\big] &= g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho, &\text{(B.13)}\\ [P^\mu, P^\nu] &= 0 &\text{(B.14)} \end{aligned}$$

which define the Poincaré algebra. A shortcut to arrive at the Lorentz algebra relation (B.12) is to calculate the generator $M^{\mu\nu}$ directly in the four-dimensional representation, where $U(\Lambda,0) = \Lambda$ is the Lorentz transformation itself:

$$U(\Lambda,0)^\alpha{}_\beta = \delta^\alpha_\beta + \tfrac{i}{2}\,\varepsilon_{\mu\nu}\,(M^{\mu\nu})^\alpha{}_\beta + \dots = \Lambda^\alpha{}_\beta = \delta^\alpha_\beta + \varepsilon^\alpha{}_\beta + \dots \tag{B.15}$$

This is solved by the tensor

$$(M^{\mu\nu})^\alpha{}_\beta = -i\,(g^{\mu\alpha}\,\delta^\nu{}_\beta - g^{\nu\alpha}\,\delta^\mu{}_\beta) \tag{B.16}$$

which satisfies the commutator relation (B.12).

We can cast the Poincaré algebra relations in a less compact but more useful form. The antisymmetric matrix $\varepsilon_{\mu\nu}$ contains the six group parameters and the antisymmetric matrix $M^{\mu\nu}$ the six generators. If we define the generator of $SO(3)$ rotations $\boldsymbol{J}$ (the angular momentum) and the generator of boosts $\boldsymbol{K}$ via

$$M^{ij} = -\varepsilon_{ijk}\,J^k \quad \Leftrightarrow \quad J^i = -\tfrac{1}{2}\,\varepsilon_{ijk}\,M^{jk}\,, \qquad M^{0i} = K^i\,, \tag{B.17}$$

then the commutator relations take the form

$$\begin{array}{lll} [J^i, J^j] = i\varepsilon_{ijk}\,J^k, & [J^i, P^j] = i\varepsilon_{ijk}\,P^k, & [P^i, P^j] = 0, \\ [J^i, K^j] = i\varepsilon_{ijk}\,K^k, & [K^i, P^j] = i\delta_{ij}\,P_0, & [J^i, P_0] = 0, \\ [K^i, K^j] = -i\varepsilon_{ijk}\,J^k, & [K^i, P_0] = iP^i, & [P^i, P_0] = 0\,. \end{array} \tag{B.18}$$

If we similarly define $\varepsilon_{ij} = -\varepsilon_{ijk}\,\phi^k$ and $\varepsilon_{0i} = s^i$, we obtain

$$\tfrac{i}{2}\,\varepsilon_{\mu\nu}\,M^{\mu\nu} = i\boldsymbol{\phi}\cdot\boldsymbol{J} + i\boldsymbol{s}\cdot\boldsymbol{K}\,. \tag{B.19}$$

$\boldsymbol{J}$ is hermitian but, because the Lorentz group is not compact, $\boldsymbol{K}$ is antihermitian for all finite-dimensional representations which prevents them from being unitary. From (B.18) we see that boosts and rotations generally do not commute unless the boost and rotation axes coincide. Moreover, P_0 (which becomes the Hamilton operator in the quantum theory) commutes with rotations and spatial translations but not with boosts and therefore the eigenvalues of $\boldsymbol{K}$ cannot be used for labeling physical states.

Casimir operators. The Casimir operators of a Lie group are those that commute with all generators and therefore allow us to label the irreducible representations. The Lorentz group has two Casimirs which are given by

$$C_1 = \tfrac{1}{2}\, M^{\mu\nu} M_{\mu\nu} = \boldsymbol{J}^2 - \boldsymbol{K}^2\,, \qquad C_2 = \tfrac{1}{2}\, \widetilde{M}^{\mu\nu} M_{\mu\nu} = 2\boldsymbol{J}\cdot\boldsymbol{K}\,. \tag{B.20}$$

The 'dual' generator is defined in analogy to Eq. (1.14): $\widetilde{M}_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\, M^{\alpha\beta}$. Using $[AB,C] = A\,[B,C] + [A,C]\,B$ it is straightforward to check that both operators commute with $M^{\mu\nu}$; they are Lorentz-*invariant*.

Unfortunately, when we turn to the full Poincaré group C_1 and C_2 do not commute with P^μ, so they are not Poincaré-invariant. In turn, $P^2 = P^\mu P_\mu$ is invariant; from Eqs. (B.13–B.14) it is easy to see that it commutes with all generators P^μ and $M^{\mu\nu}$ (for example, the contraction of (B.13) with P^μ gives zero). P^2 is therefore a Casimir operator of the Poincaré group. The second Casimir is the square $W^2 = W^\mu W_\mu$ of the Pauli-Lubanski vector

$$W_\mu = -\frac{1}{2}\,\varepsilon_{\mu\rho\sigma\lambda}\, M^{\rho\sigma} P^\lambda\,. \tag{B.21}$$

Since W^μ is a four-vector, W^2 is Lorentz-invariant and must commute with $M^{\mu\nu}$. W^μ commutes with the momentum operator because of Eq. (B.13), $[P^\mu, W^\nu] = 0$, and therefore also $[P^\mu, W^2] = 0$. Hence, both P^2 and W^2 are not only Lorentz- but also Poincaré-invariant. Written in components, the Pauli-Lubanski vector has the form

$$W_0 = \boldsymbol{P}\cdot\boldsymbol{J}\,, \qquad \boldsymbol{W} = P_0\,\boldsymbol{J} + \boldsymbol{P}\times\boldsymbol{K}\,. \tag{B.22}$$

Working out W^2 in generality is a bit cumbersome, but for $P^2 = m^2 > 0$ we can define a rest frame where $\boldsymbol{P} = 0$. In that frame one has $W_0 = 0$, $\boldsymbol{W} = m\,\boldsymbol{J}$ and $W^2 = -m^2\boldsymbol{J}^2$. The eigenvalues of $\boldsymbol{J}^2$ in the rest frame are $j(j+1)$, but since W^2 is Poincaré-invariant, so must be j. Here lies the origin of *spin*: from the point of view of the Poincaré group, the mass m and spin j are the only Poincaré-invariant quantum numbers that we can assign to a physical state.

We can derive this in another way so that also the connection with the Casimior operators (B.20) of the Lorentz group becomes more transparent. Define the transverse projection of $M^{\mu\nu}$ with respect to P:

$$M_\perp^{\mu\nu} := T^{\mu\alpha}\, T^{\nu\beta} M_{\alpha\beta} \qquad \text{with} \quad T^{\mu\nu} = g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\,. \tag{B.23}$$

Because the components P^μ commute among themselves and also with P^2, they also commute with the transverse projector,

$$[P^\mu, M_\perp^{\rho\sigma}] = [P^\mu, T^{\rho\alpha}\, T^{\sigma\beta} M_{\alpha\beta}] = T^{\rho\alpha}\, T^{\sigma\beta}\, [P^\mu, M_{\alpha\beta}] \stackrel{\text{(B.13)}}{=} 0\,, \tag{B.24}$$

and the commutator relations (B.12–B.14) become

$$\begin{aligned} i\big[M_\perp^{\mu\nu}, M_\perp^{\rho\sigma}\big] &= T^{\mu\sigma} M_\perp^{\nu\rho} + T^{\nu\rho} M_\perp^{\mu\sigma} - T^{\mu\rho} M_\perp^{\nu\sigma} - T^{\nu\sigma} M_\perp^{\mu\rho}\,, \\ \big[P^\mu, M_\perp^{\rho\sigma}\big] &= 0\,, \\ [P^\mu, P^\nu] &= 0\,. \end{aligned} \tag{B.25}$$

The square of $M_\perp^{\mu\nu}$ is now indeed Poincaré-invariant because it commutes not only with $M_\perp^{\mu\nu}$ but also with P^μ. To establish the relation with W^2, one can derive[1]

$$\begin{aligned} M_\perp^{\mu\nu} &= -\frac{1}{P^2}\,\varepsilon^{\mu\nu\alpha\beta}P_\alpha\,W_\beta\,, \\ \widetilde{M}_\perp^{\mu\nu} &= \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}(M_\perp)_{\alpha\beta} = \frac{1}{P^2}\,(P^\mu W^\nu - P^\nu W^\mu)\,, \end{aligned} \tag{B.26}$$

from where it follows that

$$W^2 = -\frac{P^2}{2}\,M_\perp^{\mu\nu}(M_\perp)_{\mu\nu}\,, \qquad \widetilde{M}_\perp^{\mu\nu}(M_\perp)_{\mu\nu} = 0\,. \tag{B.27}$$

W^2 is therefore the analogue of C_1 from the Lorentz group whereas the remaining possible Casimir vanishes identically. Along the same lines one obtains the relation

$$[W^\mu, W^\nu] = -iP^2 M_\perp^{\mu\nu} = i\varepsilon^{\mu\nu\alpha\beta}P_\alpha\,W_\beta \tag{B.28}$$

that will become useful later. From the $1/P^2$ factors in the denominators of these expressions we also see that the massless case $P^2 = 0$ will be special, cf. Sec. B.3.

B.2 Representations of the Lorentz group

Reducible vs. irreducible representations. Let's work out the irreducible representations of the Lorentz group. The discussion is similar to that in App. A for $SU(N)$ except for some additional complications due to the richer structure of the group. A Lorentz tensor of rank n is defined by the transformation law

$$(T')^{\mu\nu\dots\tau} = \underbrace{\Lambda^\mu{}_\alpha\,\Lambda^\nu{}_\beta\dots\Lambda^\tau{}_\lambda}_{n\text{ times}}\,T^{\alpha\beta\dots\lambda}\,, \tag{B.29}$$

so we can always construct the representation matrices $\Lambda^\mu{}_\alpha\,\Lambda^\nu{}_\beta\cdots$ of the Lorentz transformation as the outer product $\mathbf{4}\otimes\mathbf{4}\otimes\cdots$ of the 4-dimensional defining representation Λ. However, these representations are not irreducible. Take for example the 4×4 tensor $T^{\mu\nu}$, which has in principle 16 components. Its trace, its antisymmetric component, and its symmetric and traceless part,

$$S = T^\alpha{}_\alpha, \qquad A^{\mu\nu} = \tfrac{1}{2}\,(T^{\mu\nu} - T^{\nu\mu}), \qquad S^{\mu\nu} = \tfrac{1}{2}\,(T^{\mu\nu} + T^{\nu\mu}) - \tfrac{1}{4}\,g^{\mu\nu}\,S, \tag{B.30}$$

do not mix under Lorentz transformations: an (anti-) symmetric tensor is still (anti-) symmetric after the transformation, and the trace S is Lorentz-invariant. The trace is one-dimensional, the antisymmetric part defines a 6-dimensional subspace, and the symmetric and traceless part a 9-dimensional subspace, so we have the decomposition $\mathbf{4}\otimes\mathbf{4} = \mathbf{1}\oplus\mathbf{6}\oplus\mathbf{9}$.

[1] Use the properties that $\varepsilon_{\mu\rho\sigma\lambda}\,M^{\rho\sigma}P^\lambda = \varepsilon_{\mu\rho\sigma\lambda}\,M_\perp^{\rho\sigma}P^\lambda$ in the definition of W^μ, that P^λ commutes with $M_\perp^{\rho\sigma}$ and W^μ, and insert the identity $\varepsilon_{\mu\alpha\beta\lambda}\,\varepsilon^\mu{}_{\rho\sigma\tau}\,P^\lambda P^\tau = -P^2\,(T_{\alpha\rho}T_{\beta\sigma} - T_{\alpha\sigma}T_{\beta\rho})$. Note that the $\varepsilon-$tensor switches sign when lowering or raising spatial indices; $\varepsilon_{\mu\nu\alpha\beta} = 1$ and $\varepsilon^{\mu\nu\alpha\beta} = -1$ for an even permutation of the indices (0123).

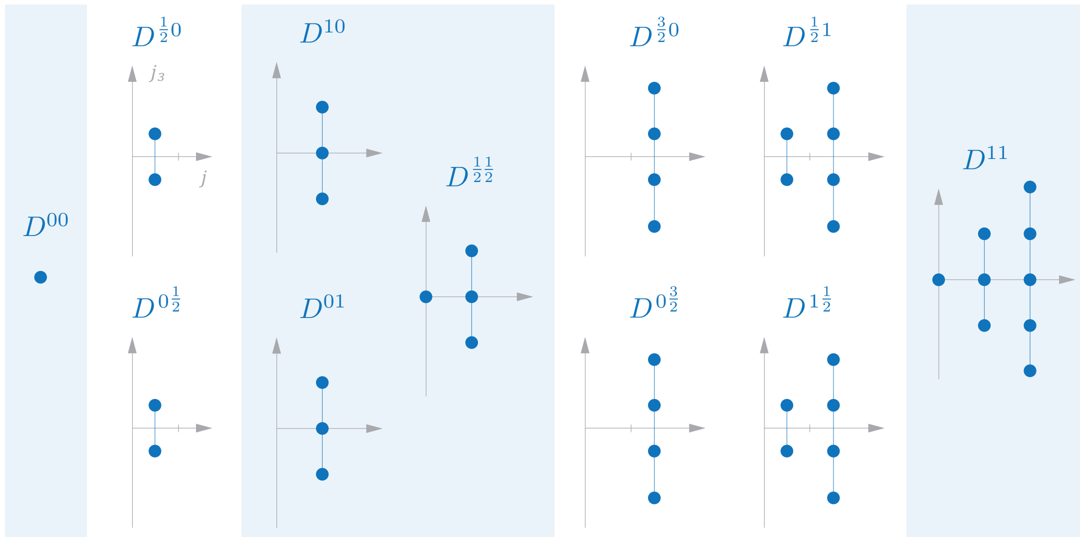

FIGURE B.2: Multiplets of the Lorentz group: tensor (shaded) vs. spinor representations.

Is there a simple way to classify the irreducible representations of the Lorentz group? If we define

$$\boldsymbol{A} = \tfrac{1}{2}\,(\boldsymbol{J} - i\boldsymbol{K}), \qquad \boldsymbol{B} = \tfrac{1}{2}\,(\boldsymbol{J} + i\boldsymbol{K}) \tag{B.31}$$

and calculate their commutator relations using Eq. (B.18), we obtain two copies of an $SU(2)$ algebra with hermitian generators A_i and B_i:

$$[A_i, A_j] = i\varepsilon_{ijk}\,A_k\,, \qquad [B_i, B_j] = i\varepsilon_{ijk}\,B_k\,, \qquad [A_i, B_j] = 0\,. \tag{B.32}$$

The two Casimir operators $\boldsymbol{A}^2$ and $\boldsymbol{B}^2$ are linear combinations of Eq. (B.20) with eigenvalues $a\,(a+1)$ and $b\,(b+1)$, hence there are two quantum numbers $a,\, b = 0,\, \frac{1}{2},\, 1, \dots$ to label the multiplets. We will denote the irreducible representation matrices by

$$D(\Lambda) = e^{\frac{i}{2}\,\omega_{\mu\nu}\,M^{\mu\nu}} = e^{i\boldsymbol{\phi}\cdot\boldsymbol{J} + i\boldsymbol{s}\cdot\boldsymbol{K}}\,, \qquad M^{ij} = -\varepsilon_{ijk}\,J^k\,, \qquad M^{0i} = K^i\,, \tag{B.33}$$

where in an n-dimensional representation $D(\Lambda)$, $M^{\mu\nu}$, $\boldsymbol{J}$ and $\boldsymbol{K}$ are $n \times n$ matrices. The generators $M^{\mu\nu}$ are not hermitian because they contain the boost generators, and therefore the representation matrices are not unitary. Their dimension is

$$D^{ab} = (2a+1)(2b+1), \tag{B.34}$$

which leads to

$$D^{00} = \mathbf{1}\,, \quad \begin{array}{l} D^{\frac{1}{2}0} = \mathbf{2} \\ D^{0\frac{1}{2}} = \mathbf{2} \end{array}, \quad \begin{array}{l} D^{10} = \mathbf{3} \\ D^{01} = \mathbf{3} \end{array}, \quad D^{\frac{1}{2}\frac{1}{2}} = \mathbf{4}\,, \quad \dots \quad D^{11} = \mathbf{9}\,, \quad \dots \tag{B.35}$$

The generator of rotations is $\boldsymbol{J} = \boldsymbol{A} + \boldsymbol{B}$, so we can use the $SU(2)$ angular momentum addition rules to construct the states within each multiplet: the states come with all possible spins $j = |a-b| \dots a+b$, where j_3 goes from $-j$ to j, see Fig. B.2.

Tensor representations. Let's first discuss the 'tensor representations' where $a+b$ is integer (the shaded multiplets in Fig. B.2). These are the actual irreducible representations of the Lorentz group that can be constructed via Eq. (B.29):

- **Trivial representation $D^{00} = \mathbf{1}$:** here the generator is $M^{\mu\nu} = 0$ and the representation matrix is 1. This is how Lorentz scalars transform.

- **Antisymmetric representation:** the 6-dimensional antisymmetric part $A^{\mu\nu}$ of a 4×4 tensor belongs here. It is the adjoint representation because its dimension is the same as the number of generators. If $A^{\mu\nu}$ is real, it is also irreducible; if it is complex it can be further decomposed into a self-dual (D^{10}) and an anti-self-dual representation (D^{01}), depending on the sign of the condition $A^{\mu\nu} = \pm\frac{i}{2}\,\varepsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}$. In Euclidean space $A^{\mu\nu}$ is always reducible and therefore the antisymmetric representation has the form $D^{10}\oplus D^{01}$.

- **Vector representation $D^{\frac{1}{2}\frac{1}{2}} = \mathbf{4}$:** The four-dimensional vector representation plays a special role because the transformation matrix is Λ itself, and it can be used to construct all further (reducible) tensor representations according to Eq. (B.29). The transformation matrices act on four-vectors, for example the space-time coordinate x^μ or the four-momentum p^μ, and they are irreducible because Λ mixes all components of the four-vector. The generator $M^{\mu\nu}$ has the form of Eq. (B.16).

- **Tensor representation $D^{11} = \mathbf{9}$:** This is where the 9-dimensional symmetric and traceless part $S^{\mu\nu}$ of a 4×4 tensor belongs.

The Lorentz group has two invariant tensors $g^{\mu\nu}$ and $\varepsilon^{\mu\nu\alpha\beta}$ which transform as

$$\begin{aligned} g'^{\mu\nu} &= \Lambda^\mu{}_\alpha\,\Lambda^\nu{}_\beta\,g^{\alpha\beta} = g^{\mu\nu}\,, \\ \varepsilon'^{\mu\nu\rho\sigma} &= \Lambda^\mu{}_\alpha\,\Lambda^\nu{}_\beta\,\Lambda^\rho{}_\gamma\,\Lambda^\sigma{}_\delta\,\varepsilon^{\alpha\beta\gamma\delta} = (\det\Lambda)\,\varepsilon^{\mu\nu\rho\sigma}\,. \end{aligned} \tag{B.36}$$

$g^{\mu\nu}$ is a scalar and $\varepsilon^{\mu\nu\alpha\beta}$ is a pseudoscalar since it is odd under parity ($\det\Lambda = -1$). Their (anti-) symmetry can be exploited to construct the irreducible components of higher-rank tensors. For example, higher antisymmetric tensors in four dimensions become simple because we cannot antisymmetrize over more than four indices. $A^{\mu\nu\rho}$ has 4 components; they can be rearranged into a four-vector $\varepsilon_{\alpha\mu\nu\rho}\,A^{\mu\nu\rho}$ that transforms under the vector representation. $A^{\mu\nu\rho\sigma}$ has only one independent component A^{0123} that can be combined into the pseudoscalar $\varepsilon_{\mu\nu\rho\sigma}\,A^{\mu\nu\rho\sigma}$, and $A^{\mu\nu\rho\sigma\tau} = 0$.

Spinor representations. The analysis also produces spinor representations where $a+b$ is half-integer. These are not representations of the Lorentz group itself but rather *projective* representations, where instead of $D(\Lambda')\,D(\Lambda) = D(\Lambda'\Lambda)$ one has

$$D(\Lambda')\,D(\Lambda) = e^{i\varphi(\Lambda',\Lambda)}\,D(\Lambda'\Lambda)\,, \tag{B.37}$$

with a phase that depends on Λ and Λ'. In our case, $e^{i\varphi} = \pm 1$ and so the projective representations are double-valued: one can find two representation matrices $\pm D(\Lambda)$ that belong to the same Λ. However, both of them are physically equivalent and therefore the representations in Fig. B.2 are all relevant.

The origin of this behavior is that the Lorentz group, and in particular its subgroup $SO(3)$, is not simply connected. The projective representations of a group correspond to the representations of its universal covering group: it has the same Lie algebra, which reflects the property of the group close to the identity, but it is simply connected. In the same way as $SU(2)$ is the double cover of $SO(3)$, the double cover of $SO(3,1)^\uparrow$ is the group $SL(2,\mathbb{C})$. It is the set of complex 2×2 matrices with unit determinant and, like the Lorentz group, it also depends on six real parameters. A double-valued projective representation of $SO(3,1)^\uparrow$ corresponds to a single-valued representation of $SL(2,\mathbb{C})$. Similarly, the double cover of the *Euclidean* Lorentz group $SO(4)$ is $SU(2)\times SU(2)$; these are the representations that we actually derived in Fig. B.2. Hence we arrive at another type of chiral symmetry, labeled by the Casimir eigenvalues a (left-handed) and b (right-handed): representations with $a=0$ or $b=0$ have definite chirality, whereas those with $a=b$ are called non-chiral. Here are some of the lowest-dimensional irreducible spinor representations:

- **Fundamental representation:** $D^{\frac{1}{2}0}$ and $D^{0\frac{1}{2}}$ have both dimension two and carry spin $j={}^1\!/_2$. They are the (anti-) fundamental representations because all other representations can be built from them. The generators are $\boldsymbol{A}=\frac{\boldsymbol{\sigma}}{2}$ and $\boldsymbol{B}=0$ for the left-handed representation and vice versa for the right-handed one, where σ_i are the Pauli matrices, and hence the spin and boost generators become
$$D^{\frac{1}{2}0}: \;\; \boldsymbol{J}=\frac{\boldsymbol{\sigma}}{2}\,, \;\; \boldsymbol{K}=i\frac{\boldsymbol{\sigma}}{2}\,, \qquad D^{0\frac{1}{2}}: \;\; \boldsymbol{J}=\frac{\boldsymbol{\sigma}}{2}\,, \;\; \boldsymbol{K}=-i\frac{\boldsymbol{\sigma}}{2}\,. \tag{B.38}$$
The representation matrices are complex 2×2 matrices $\in SL(2,\mathbb{C})$, and the corresponding spinors are left- and right-handed Weyl spinors ψ_L, ψ_R.

- **Dirac (bispinor) representation $D^{\frac{1}{2}0}\oplus D^{0\frac{1}{2}}$:** Under a parity transformation, the rotation generators are invariant whereas the boost generators change their sign: $\boldsymbol{J}\to\boldsymbol{J}$, $\boldsymbol{K}\to-\boldsymbol{K}$. Therefore, parity exchanges $\boldsymbol{A}\leftrightarrow\boldsymbol{B}$ and transforms the two fundamental representations into each other, and a theory that is invariant under parity must necessarily include both doublets. This is the reason why spin-${}^1\!/_2$ fermions are treated as four-dimensional Dirac spinors ψ_α, which can be constructed as the direct sums of left- and right-handed Weyl spinors:
$$\boldsymbol{J}=\begin{pmatrix}\boldsymbol{\sigma}/2 & 0\\ 0 & \boldsymbol{\sigma}/2\end{pmatrix}=\frac{\boldsymbol{\Sigma}}{2}\,, \quad \boldsymbol{K}=\begin{pmatrix}i\boldsymbol{\sigma}/2 & 0\\ 0 & -i\boldsymbol{\sigma}/2\end{pmatrix}, \quad \psi=\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}. \tag{B.39}$$
The resulting generator $M^{\mu\nu}=-\frac{i}{4}\,[\gamma^\mu,\gamma^\nu]$ satisfies again the Lorentz algebra relation. The Dirac spinors transform under the four-dimensional representation matrices: $\psi'=D(\Lambda)\,\psi$, $\overline{\psi}'=\overline{\psi}\,D(\Lambda)^{-1}$. Therefore, a bilinear $\overline{\psi}\psi$ is Lorentz-invariant, $\overline{\psi}\gamma^\mu\psi$ transforms like a vector because $D(\Lambda)^{-1}\,\gamma^\mu D(\Lambda)=\Lambda^\mu{}_\nu\gamma^\nu$, etc.

- **Rarita-Schwinger representation:** The same point would in principle apply to spin-$\frac{3}{2}$ fermions in the (eight-dimensional) $D^{\frac{3}{2}0}\oplus D^{0\frac{3}{2}}$ representation, but it is more convenient to construct them as Rarita-Schwinger vector-spinors ψ^μ_α via
$$D^{\frac{1}{2}\frac{1}{2}}\otimes(D^{\frac{1}{2}0}\oplus D^{0\frac{1}{2}})=(D^{\frac{1}{2}0}\oplus D^{\frac{1}{2}1})\oplus(D^{0\frac{1}{2}}\oplus D^{1\frac{1}{2}})\,, \tag{B.40}$$
which in turn requires additional constraints to single out the spin-$\frac{3}{2}$ subspace.

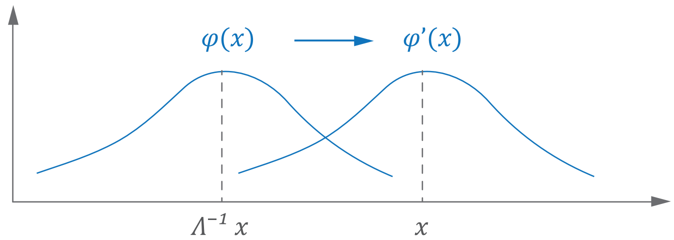

FIGURE B.3: Visualization of $\varphi'(x) = \varphi(\Lambda^{-1}x)$. Compare this with quantum mechanics: if $\boldsymbol{x} \to R\boldsymbol{x}$ and $\varphi \to U\varphi$, then $\langle \boldsymbol{x}|U\varphi\rangle = \langle R^{-1}\boldsymbol{x}|\varphi\rangle$, or equivalently: $\varphi(\boldsymbol{x}) \to U\varphi(\boldsymbol{x}) = \varphi(R^{-1}\boldsymbol{x})$.

This last example may seem a bit contrived, but remember that from the perspective of the Poincaré group only the Casimirs P^2 and W^2 are relevant. For a massive particle the eigenvalues of W^2 in the rest frame coincide with j, but since W^2 is Poincaré-invariant, all properties associated with j hold in general. Therefore, the multiplet assignment D^{ab} in Fig. B.2 is strictly speaking meaningless because the only quantity that really matters is the spin content j: a particle with spin $j = \frac{1}{2}$ has two spin polarizations, a spin-1 particle three, and so on.

In the nonrelativistic limit where Lorentz transformations reduce to spatial rotations, the multiplets in Fig. B.2 are no longer irreducible but we can decompose them with respect to $SO(3)$ (or its universal cover $SU(2)$). For example, a four-vector $V^\mu = (V^0, \boldsymbol{V})$ defines an irreducible representation of the Lorentz group, but from the point of view of the $SO(3)$ subgroup it is reducible ($\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$) because V^0 is invariant under spatial rotations (it has $j = 0$), whereas the three spatial components form an irreducible representation with $j = 1$. Similarly, the symmetric and traceless part of a 4×4 tensor is reducible: $\mathbf{9} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$.

B.3 Poincaré invariance in field theories

Field representations. So far we have only considered the Lorentz transformations of spacetime-independent quantities (scalars, vectors, spinors etc.). They transform generically as $\varphi_i' = D_{ij}(\Lambda)\,\varphi_j$, where i and j are the matrix indices in the given representation. When we consider fields $\varphi_i(x)$, the transformation $x' = \Lambda x$ must also act on the spacetime argument:

$$\varphi_i'(x) = D_{ij}(\Lambda)\,\varphi_j(\Lambda^{-1}x) \quad \Leftrightarrow \quad \varphi_i'(x') = D_{ij}(\Lambda)\,\varphi_j(x)\,. \tag{B.41}$$

The appearance of Λ^{-1} is consistent with the usual symmetry operations in quantum mechanics, cf. Fig. B.3. We can now define two types of infinitesimal transformations. The first is the same as before and expresses the 'change in perspective':

$$\delta\varphi_i = \varphi_i'(x') - \varphi_i(x) = \frac{i}{2}\,\varepsilon_{\mu\nu}\,(M_\mathrm{S}^{\mu\nu})_{ij}\,\varphi_j(x)\,, \tag{B.42}$$

with the finite-dimensional matrix representation of the generator $M^{\mu\nu}$ (we added the subscript S for *spin* to distinguish it from what comes next). For example, a scalar field $\varphi'(x') = \varphi(x)$ is Lorentz-invariant and has $\delta\varphi = 0$. On the other hand, when we want to measure how the *functional form* of the field changes at the position x (see again Fig. B.3), we have to work out

$$\delta_0\varphi_i = \varphi_i'(x) - \varphi_i(x) = \varphi_i'(x' - \delta x) - \varphi_i(x) = \delta\varphi_i - \delta x_\mu\, \partial^\mu \varphi_i\,. \tag{B.43}$$

The infinitesimal Lorentz transformation has the form $\delta x_\mu = \varepsilon_{\mu\nu}\, x^\nu$, and therefore

$$-\,\delta x_\mu\, \partial^\mu \varphi_i = -\varepsilon_{\mu\nu}\, x^\nu \partial^\mu \varphi_i = \frac{i}{2}\,\varepsilon_{\mu\nu}\, \underbrace{\left[-i\,(x^\mu \partial^\nu - x^\nu \partial^\mu)\right]}_{=:\,M_\mathrm{L}^{\mu\nu}}\, \varphi_i\,, \tag{B.44}$$

where $M_\mathrm{L}^{\mu\nu}$ contains the orbital angular momentum and satisfies again the Lorentz algebra relations. Before discussing it further, let's generalize this to Poincaré transformations right away. For pure translations each component of the field is a scalar:

$$\varphi_i'(x) = \varphi_i(x-a) \quad \Leftrightarrow \quad \varphi_i'(x') = \varphi_i(x)\,, \tag{B.45}$$

and hence $\delta\varphi_i = 0$ and $\delta_0\varphi_i = -a_\mu \partial^\mu \varphi_i = i a_\mu P^\mu \varphi_i$, with $P^\mu = i\partial^\mu$. The total change of the field is therefore

$$\varphi_i'(x) = \varphi_i(x) + \left[\frac{i}{2}\,\varepsilon_{\mu\nu}\,(M_\mathrm{S}^{\mu\nu} + M_\mathrm{L}^{\mu\nu}) + i a_\mu P^\mu\right]_{ij} \varphi_j(x)\,. \tag{B.46}$$

$M_\mathrm{L}^{\mu\nu}$ and P^μ are differential operators that satisfy the Poincaré algebra relations when applied to $\varphi_i(x)$. They are diagonal in i,j whereas the spin matrix $M_\mathrm{S}^{\mu\nu}$ depends on the representation of the field. In the same way as $M^{\mu\nu} = M_\mathrm{S}^{\mu\nu} + M_\mathrm{L}^{\mu\nu}$, the angular momentum and boost generators extracted from Eq. (B.17) are the sums of spin and orbital angular momentum parts: $\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L}$ and $\boldsymbol{K} = \boldsymbol{K}_\mathrm{S} + \boldsymbol{K}_\mathrm{L}$, with

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{P}\,, \qquad \boldsymbol{K}_\mathrm{L} = \boldsymbol{x}\, P^0 - x^0 \boldsymbol{P}\,, \qquad P^\mu = i\partial^\mu\,. \tag{B.47}$$

Note that the boost generator is explicitly time-dependent.

Poincaré invariance of the action. The invariance of the classical action under Poincaré transformations has similar consequences as for global symmetry groups, cf. Sec. 2.1: there are conserved Noether currents, and after quantization the corresponding charges form a representation of the Poincaré algebra on the state space.

To derive the current we have to add variations of spacetime to Eq. (2.1):

$$\delta S = \underbrace{\int d^4x\, \delta_0 \mathcal{L}}_{\text{Eq. (2.1)}} + \int d^4x\, \partial_\mu \mathcal{L}\, \delta x^\mu + \int (\delta d^4 x)\, \mathcal{L} = \int d^4x \left[\delta_0\mathcal{L} + \partial_\mu(\mathcal{L}\,\delta x^\mu)\right]. \tag{B.48}$$

The first term is the same as in Eq. (2.1) except for the replacement $\delta \to \delta_0$, because it contains only the variation in the functional form of the fields. To arrive at the last expression we used $\delta d^4x = d^4x\, \partial_\mu \delta x^\mu$. The new derivative term will contribute to the current, which becomes

$$-\,\delta j^\mu = \mathcal{L}\,\delta x^\mu + \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\, \delta_0\varphi_i\,. \tag{B.49}$$

Inserting $\delta_0\varphi_i = \delta\varphi_i - \delta x_\alpha\,\partial^\alpha\varphi_i$ from Eq. (B.43), we can reexpress this in terms of $\delta\varphi_i$:

$$\delta j^\mu = \underbrace{\left[\sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\,\partial^\alpha\varphi_i - g^{\mu\alpha}\,\mathcal{L}\right]}_{=:\,T^{\mu\alpha}}\delta x_\alpha - \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\,\delta\varphi_i\,. \tag{B.50}$$

$T^{\mu\alpha}$ defines the energy-momentum tensor whose T^{00} component is the Hamiltonian density: $T^{00} = \pi_i\,\dot\varphi_i - \mathcal{L} = \mathcal{H}$. We can now derive two types of conserved currents that reflect the invariance under translations or Lorentz transformations:

- For pure translations $x \to x + a$ we have $\delta x_\alpha = a_\alpha$ and the fields are invariant, $\delta\varphi_i = 0$. Hence, the second term in (B.50) drops out and the translation current is just the energy-momentum tensor itself: $\delta j^\mu = a_\alpha\,T^{\mu\alpha}$. Translation invariance of the action entails that its divergence vanishes: $\partial_\mu\,T^{\mu\alpha} = 0$.

- For pure Lorentz transformations the group parameters are $\varepsilon_{\alpha\beta}$ and therefore

  $$\delta x_\alpha = \varepsilon_{\alpha\beta}\,x^\beta\,, \qquad \delta\varphi_i = \frac{i}{2}\,\varepsilon_{\alpha\beta}\,(M_\mathrm{S}^{\alpha\beta})_{ij}\,\varphi_j\,. \tag{B.51}$$

  Inserting this into Eq. (B.50), writing $\delta j^\mu = \frac{1}{2}\,\varepsilon_{\alpha\beta}\,m^{\mu,\alpha\beta}$, and using the antisymmetry of $\varepsilon_{\alpha\beta}$ we find the conserved current

  $$m^{\mu,\alpha\beta} = T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha + s^{\mu,\alpha\beta}\,, \quad s^{\mu,\alpha\beta} = -i\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\,(M_\mathrm{S}^{\alpha\beta})_{ij}\,\varphi_j\,, \tag{B.52}$$

  with $\partial_\mu\,m^{\mu,\alpha\beta} = 0$. The first two terms encode the orbital angular momentum and the third term is the spin current.[2]

If we substitute the explicit form of the energy-momentum tensor into Eq. (B.52) together with $P^\alpha = i\partial^\alpha$ and $M^{\mu\nu} = M_\mathrm{S}^{\mu\nu} + M_\mathrm{L}^{\mu\nu}$, we can write the two currents as

$$\begin{aligned} T^{\mu\alpha} &= -i\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\,P^\alpha\,\varphi_j - g^{\mu\alpha}\,\mathcal{L}\,, \\ m^{\mu,\alpha\beta} &= -i\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\,M_{ij}^{\alpha\beta}\,\varphi_j + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha})\,\mathcal{L}\,. \end{aligned} \tag{B.53}$$

The corresponding constants of motion, whose total time derivatives vanish, are the zero components of the currents $T^{\mu\alpha}$ and $m^{\mu,\alpha\beta}$ when integrated over d^3x:

$$\hat P^\alpha = \int d^3x\,T^{0\alpha}\,, \qquad \hat M^{\alpha\beta} = \int d^3x\,m^{0,\alpha\beta}\,. \tag{B.54}$$

In the quantum field theory they will form another representation of the Poincaré algebra that acts on the state space.

[2] An alternative form of the energy-momentum tensor is the Belinfante tensor, which is still conserved (and hence physically equivalent) but symmetric in α and β: $\Theta^{\alpha\beta} = T^{\alpha\beta} - \frac{1}{2}\,\partial_\mu\,(s^{\mu,\alpha\beta} + s^{\alpha,\beta\mu} - s^{\beta,\mu\alpha})$. To prove this, use the antisymmetry of $s^{\mu,\alpha\beta}$ in α,β and the conservation law $\partial_\mu m^{\mu,\alpha\beta} = 0$.

Dirac theory. As an example, consider a free Dirac Lagrangian $\mathcal{L} = \overline{\psi}\,(\not{P} - m)\,\psi$. The Poincaré transformation of the field is $\psi'(x') = D(\Lambda)\,\psi(x)$, where $D(\Lambda)$ has the form of Eq. (B.33) with $M_{\text{S}}^{\mu\nu} = -\frac{1}{2}\,\sigma^{\mu\nu} = -\frac{i}{4}\,[\gamma^\mu,\gamma^\nu]$. From Eq. (B.53) we have

$$\begin{array}{l} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \overline{\psi}\,i\gamma^\mu \\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0 \end{array} \quad\Rightarrow\quad \begin{array}{ll} T^{00} = \psi^\dagger P^0\,\psi - \mathcal{L}\,, & m^{0,ij} = \psi^\dagger M^{ij}\,\psi\,, \\ T^{0i} = \psi^\dagger P^i\,\psi\,, & m^{0,0i} = \psi^\dagger M^{0i}\,\psi - x^i\mathcal{L}\,, \end{array} \tag{B.55}$$

and we can read off the constants of motion ($\Sigma^i = \frac{1}{2}\,\varepsilon_{ijk}\,\sigma^{jk}$):

$$\hat{P}^0 = \int d^3x\,\overline{\psi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{P} + m)\,\psi\,, \quad \hat{\boldsymbol{P}} = \int d^3x\,\psi^\dagger\boldsymbol{P}\,\psi\,, \quad \hat{\boldsymbol{J}} = \int d^3x\,\psi^\dagger\left[\boldsymbol{x}\times\boldsymbol{P} + \frac{\boldsymbol{\Sigma}}{2}\right]\psi\,.$$

In relativistic quantum mechanics the field $\psi(x)$ is interpreted as a particle's wave function that belongs to a Hilbert space, and a Lorentz-invariant scalar product for solutions of the Dirac equation $(\not{P} - m)\,\psi = 0$ is imposed:

$$\langle\psi|\psi\rangle := \int d\sigma_\mu\,\overline{\psi}(x)\,\gamma^\mu\,\psi(x) = \int d^3x\,\psi^\dagger(x)\,\psi(x)\,. \tag{B.56}$$

It has the same value on each spacelike hypersurface σ in Minkowski space, and choosing it to be a slice at fixed time yields the second form. For solutions of the classical equations of motion the terms proportional to the Dirac Lagrangian $\mathcal{L}$ in (B.55) can be dropped and the conserved charges become the expectation values of the operators P^α and $M^{\alpha\beta}$:

$$\hat{P}^\alpha = \int d^3x\,T^{0\alpha} = \langle\psi|P^\alpha\psi\rangle\,, \qquad \hat{M}^{\alpha\beta} = \int d^3x\,m^{0,\alpha\beta} = \langle\psi|M^{\alpha\beta}\psi\rangle\,. \tag{B.57}$$

One can show that both operators P^α and $M^{\alpha\beta}$ are hermitian: $\langle\psi_1|O\,\psi_2\rangle = \langle O\,\psi_1|\psi_2\rangle$, and therefore the representation provided by Eq. (B.46) is unitary. This has become possible because, when applied to spacetime-dependent fields $\psi(x)$ that depend on a continuous and unbound variable x, the representations are now infinite-dimensional (they are differential operators). Specifically, the spin contribution to the boost generator $K_S^i = -\frac{1}{2}\,\sigma^{0i} = -\frac{i}{2}\gamma^0\gamma^i$ is still an antihermitian matrix, but its sum $\boldsymbol{K} = \boldsymbol{K}_{\text{S}} + \boldsymbol{K}_{\text{L}}$ with the differential operator $\boldsymbol{K}_L = \boldsymbol{x}\,P^0 - x^0\boldsymbol{P}$ is indeed hermitian. An analogous Lorentz-invariant scalar product for scalar fields $\phi(x)$ is

$$\langle\phi|\phi\rangle = \frac{i}{2}\int d\sigma^\mu\phi^*(x)\,\overleftrightarrow{\partial}_\mu\,\phi(x) = \frac{i}{2}\int d^3x\,\phi^*(x)\,\overleftrightarrow{\partial}_0\,\phi(x)\,, \qquad \overleftrightarrow{\partial}_\mu = \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu\,. \tag{B.58}$$

Unitary representations of the Poincaré group. Now what about the quantum field theory? A theorem by Wigner states that continuous symmetries must be implemented by unitary operators on the state space. The Lorentz group is not compact because it contains boosts, hence all unitary representations must be infinite-dimensional. This is realized in the quantum field theory: the fields $\varphi_i(x)$ become operators on the Fock space, and the constants of motion in Eq. (B.54) are hermitian operators that define a unitary representation of the Poincaré algebra on the state space:

$$U(\Lambda,a) = e^{\frac{i}{2}\varepsilon_{\mu\nu}\,\hat{M}^{\mu\nu}}\,e^{ia_\mu\hat{P}^\mu} = 1 + \tfrac{i}{2}\,\varepsilon_{\mu\nu}\,\hat{M}^{\mu\nu} + ia_\mu\hat{P}^\mu + \ldots \tag{B.59}$$

What is the irreducible state space? One of the axioms of quantum field theory is that the vacuum is the only Poincaré-invariant state: $U(\Lambda,a)\,|0\rangle = |0\rangle$.[3] The Poincaré group has two Casimir operators P^2 and W^2 (we dropped the hats again). With $[P^\mu, W^\nu]=0$ and Eq. (B.28) there are at most six operators that commute with each other and can be used to label the eigenstates: P^μ, W^2, and one component of the Pauli-Lubanski vector W^μ. Considering one-particle states, this allows us to work with eigenstates of the momentum operator:

$$P^\mu|p,\dots\rangle = p^\mu|p,\dots\rangle \quad \Rightarrow \quad U(1,a)\,|p,\dots\rangle = e^{ia\cdot p}\,|p,\dots\rangle\,, \tag{B.60}$$

where the dots are the remaining quantum numbers.

To construct the general form of the representation, let's start with a massive particle at rest. We denote the rest-frame momentum by $\mathring{p}=(m,\mathbf{0})$. The group that leaves a given choice of momentum p^μ invariant is called the *little* group; its generators are the independent components of the Pauli-Lubanski vector. Since rotations leave the rest-frame momentum $\mathring{p}^\mu$ invariant, the independent components are the generators J^i, cf. Eq. (B.22), and the little group is $SO(3)$ — or actually $SU(2)$ because we want to include spinor representations as well. Hence these operators take the form $P^2=m^2$, $W^2=-m^2\boldsymbol{J}^2$ and $W^3 = mJ^3$, where J^3 has eigenvalue σ and the eigenvectors are

$$P^\mu\,|\mathring{p},j\sigma\rangle = \mathring{p}^\mu\,|\mathring{p},j\sigma\rangle\,, \qquad \boldsymbol{J}^2\,|\mathring{p},j\sigma\rangle = j(j+1)\,|\mathring{p},j\sigma\rangle\,, \qquad J^3\,|\mathring{p},j\sigma\rangle = \sigma\,|\mathring{p},j\sigma\rangle\,. \tag{B.61}$$

This is the standard angular momentum algebra, and therefore rotations R are represented by the unitary matrices $\mathcal{D}^j(R)$ with $\sigma\in[-j,j]$:

$$U(R,0)\,|\mathring{p},j\sigma\rangle = \sum_{\sigma'} \mathcal{D}^j_{\sigma'\sigma}(R)\,|\mathring{p},j\sigma\rangle\,. \tag{B.62}$$

On the other hand, a boost from $\mathring{p}$ to p, which we denote by $\mathsf{L}(p)$, will have the effect

$$U(\mathsf{L}(p),0)\,|\mathring{p},j\sigma\rangle = |p,j\sigma\rangle\,. \tag{B.63}$$

With that we have everything in place to apply a general Lorentz transformation $U(\Lambda,0)$ to a state vector $|p,j\sigma\rangle$:

$$\begin{aligned} U(\Lambda,0)\,|p,j\sigma\rangle &= U(\Lambda,0)\,U(\mathsf{L}(p),0)\,|\mathring{p},j\sigma\rangle \\ &= U(\mathsf{L}(\Lambda p)\,\underbrace{\mathsf{L}^{-1}(\Lambda p)\,\Lambda\,\mathsf{L}(p)}_{=:R_W},0)\,|\mathring{p},j\sigma\rangle. \end{aligned} \tag{B.64}$$

The Wigner rotation $R_W(\Lambda,p)$ is a pure rotation that leaves the rest-frame vector invariant, because $\mathsf{L}(p)\,\mathring{p}=p$ entails $R_W\,\mathring{p} = \mathsf{L}^{-1}(\Lambda p)\,\Lambda p = \mathring{p}$. Think of it as a journey along the mass shell that leads back to the starting point: $\mathring{p}\to p\to\Lambda p\to\mathring{p}$. This is extremely helpful because from Eq. (B.62) we know how rotations act on the state space, and in combination with Eqs. (B.63) and (B.60) we arrive at the final result:

$$U(\Lambda,a)\,|\,p,j\sigma\rangle = e^{ia\cdot(\Lambda p)}\sum_{\sigma'}\mathcal{D}^{(j)}_{\sigma'\sigma}(R_W)\,\big|\,\Lambda p,j\sigma'\big\rangle\,. \tag{B.65}$$

[3] Actually, translation invariance and uniqueness of the vacuum is sufficient to prove this.

That the representation is unitary can be seen from the scalar product:

$$\langle p, j\sigma \,|\, U^\dagger(\Lambda, a)\, U(\Lambda, a)\,|\, p', j'\sigma'\rangle = \langle \Lambda p, j\sigma \,|\, \Lambda p', j'\sigma'\rangle = \langle p, j\sigma \,|\, p', j\sigma\rangle\,. \tag{B.66}$$

In the first equality the representation matrices $\mathcal{D}^j$ and the phases $e^{ia\cdot(\Lambda p)}$ cancel each other, and the second equality holds because $\langle\lambda|\lambda'\rangle = (2\pi)^3\, 2E_{\boldsymbol{p}}\,\delta(\boldsymbol{p}-\boldsymbol{p}')\,\delta_{\lambda\lambda'}$ is Lorentz-invariant. Hence, we have a unitary implementation of the Poincaré group in the quantum field theory, as required by Wigner's theorem.

Massless particles. Massless particles with $P^2 = 0$ do not have a rest frame, but the construction of the irreducible representations is very similar. Here we can choose $\mathring{p} = \omega\,(1, \boldsymbol{n})$ to be some momentum on the light cone, and the little group $SO(2)$ (or equivalently $U(1)$) consists of the rotations around the momentum axis $\boldsymbol{n}$. The generator is the helicity $\boldsymbol{J}\cdot\boldsymbol{n}$, whose eigenvalue λ can be shown to be quantized: $\lambda = 0, \pm\frac{1}{2}, \pm 1$, etc. Hence, massless particles have no spin but only two components of the helicity that are measurable.[4] The steps are the same as before, with the Wigner rotation R_W defined as in Eq. (B.64) except that $\mathcal{D}(R_W) = e^{i\lambda\,\theta(\Lambda,p)}$ is just a phase:

$$U(\Lambda, a)\,|\, p, \lambda\rangle = e^{ia\cdot(\Lambda p)}\,\mathcal{D}(R_W)\,|\,\Lambda p, \lambda\rangle\,. \tag{B.67}$$

In principle this also implies that the helicity is Poincaré-invariant and $\pm\lambda$ corresponds to different species of particles. However, the same reasoning that required us earlier to implement spinors with both chiralities also applies here: $\boldsymbol{J}\cdot\boldsymbol{n}$ is a pseudoscalar and changes sign under parity, and a theory that conserves parity must treat both helicity states symmetrically. A combined representation of the Poincaré group and parity identifies $\pm\lambda$ with the two polarizations of the same particle (e.g. the photon in QED).

Transformation of field operators and $n-$point functions. Field operators transform in the same way as in Eq. (B.41) if we insert $\varphi_i' = U(\Lambda, a)^{-1}\varphi_i\, U(\Lambda, a)$. Shuffling things around between the left and right, it is more convenient to write

$$U(\Lambda, a)\,\varphi_i(x)\, U(\Lambda, a)^{-1} = D(\Lambda)^{-1}_{ij}\,\varphi_j(\Lambda x + a)\,. \tag{B.68}$$

As before, the field operator $\varphi_i(x)$ belongs to some finite-dimensional multiplet of the Lorentz group and $D(\Lambda)$ is the corresponding spin matrix of the Lorentz transformation. For example, we have $D(\Lambda) = 1$ for a scalar field, $D(\Lambda) = \Lambda$ for a vector field or $D(\Lambda) = \exp(-\frac{i}{4}\,\varepsilon_{\mu\nu}\,\sigma^{\mu\nu})$ for a Dirac spinor field.

Matrix elements are Lorentz-covariant and transform under these matrix representations. Take for example a scalar Bethe-Salpeter wave function of two scalar fields, $\chi(x_1, x_2, p) = \langle 0|\,\mathsf{T}\,\varphi(x_1)\,\varphi(x_2)\,|p\rangle$. In that case Eqs. (B.65) and (B.68) simplify to

$$U_X = U(0, X): \qquad U_X\,|p\rangle = e^{ip\cdot X}|p\rangle\,, \qquad U_X\,\varphi(x)\,U_X^{-1} = \varphi(x+X)\,, \tag{B.69}$$

$$U_\Lambda = U(\Lambda, 0): \qquad U_\Lambda\,|p\rangle = |\Lambda p\rangle\,, \qquad U_\Lambda\,\varphi(x)\,U_\Lambda^{-1} = \varphi(\Lambda x)\,. \tag{B.70}$$

[4]In fact, the Pauli-Lubanski operator W^μ has *three* independent components in the massless case: the helicity $\boldsymbol{J}\cdot\boldsymbol{n}$ and two components perpendicular to $\boldsymbol{n}$. One can show, however, that the transverse components lead to representations with continuous spin $W^2 > 0$, which are not observed in nature and must be excluded. Evaluated on the helicity states, the spin is zero: $W^2 = 0$.

Translation invariance has the consequence that only the relative coordinate $x := x_1 - x_2$ is relevant because the dependence on the total position $X := \frac{x_1+x_2}{2}$ can only enter through a phase:

$$\begin{aligned} \chi(x_1, x_2, p) &= \langle 0|\,\mathsf{T}\,\varphi(x_1)\,\varphi(x_2)\,|p\rangle = \langle 0|\,\mathsf{T}\,\varphi(X+\tfrac{x}{2})\,\varphi(X-\tfrac{x}{2})\,|p\rangle \\ &= \langle 0|\,\mathsf{T}\,U_X\,\varphi(\tfrac{x}{2})\,U_X^{-1}\,U_X\,\varphi(-\tfrac{x}{2})\,U_X^{-1}\,|p\rangle \\ &= \langle 0|\,\mathsf{T}\,\varphi(\tfrac{x}{2})\,\varphi(-\tfrac{x}{2})\,|p\rangle\,e^{-ip\cdot X} = \chi(x,p)\,e^{-ip\cdot X}, \end{aligned} \tag{B.71}$$

where we used translation invariance of the vacuum. In turn, the wave function $\chi(x, p)$ is Lorentz-invariant:

$$\begin{aligned} \chi(x, p) &= \langle 0|\,\mathsf{T}\,\varphi(\tfrac{x}{2})\,\varphi(-\tfrac{x}{2})\,|p\rangle \\ &= \langle 0|\,\mathsf{T}\,U_\Lambda^{-1}\,U_\Lambda\,\varphi(\tfrac{x}{2})\,U_\Lambda^{-1}\,U_\Lambda\,\varphi(-\tfrac{x}{2})\,U_\Lambda^{-1}\,U_\Lambda\,|p\rangle \\ &= \langle 0|\,\mathsf{T}\,\varphi(\tfrac{\Lambda x}{2})\,\varphi(-\tfrac{\Lambda x}{2})\,|\Lambda p\rangle = \chi(\Lambda x, \Lambda p). \end{aligned} \tag{B.72}$$

The time ordering commutes with the transformation because the sign of $(x_1 - x_2)^0$ is invariant under $ISO(3,1)^\uparrow$. If we set $p = 0$ in the first equation we also see that translation invariance for the two-point function $\langle 0|\,\mathsf{T}\,\varphi(x_1)\,\varphi(x_2)\,|0\rangle$ (and generally for any $n-$point function) means that the total coordinate drops out completely.

We can repeat the steps in Eq. (B.72) for matrix elements that contain fields in some general Lorentz representation. For example, for a $q\bar{q}$ vector Green function $G^\mu(x, x_1, x_2) = \langle 0|\,\mathsf{T}\,j^\mu(x)\,\psi(x_1)\,\overline{\psi}(x_2)\,|0\rangle$ we obtain

$$G^\mu(x, x_1, x_2) = (\Lambda^{-1})^\mu{}_\nu\,D^{-1}(\Lambda)\,G^\nu(\Lambda x, \Lambda x_1, \Lambda x_2)\,D(\Lambda)\,, \tag{B.73}$$

where $D(\Lambda)$ is again the transformation matrix for Dirac spinors coming from the quark fields. The analogous equation in momentum space,

$$G^\mu(p, q) = (\Lambda^{-1})^\mu{}_\nu\,D^{-1}(\Lambda)\,G^\nu(\Lambda p, \Lambda q)\,D(\Lambda)\,, \tag{B.74}$$

can be immediately verified for the various tensor structures that contribute to the three-point function: γ^μ, p^μ, $p^\mu\,\not{p}$, $\gamma^\mu\not{p}$, etc. In covariant equations where these objects are combined in loop integrals (perturbation series, Dyson-Schwinger equations, etc.), all internal representation matrices cancel each other and only the overall factors of the diagrams remain, which can be factored out. It is then not necessary to perform explicit Lorentz transformations when changing the frame; one can simply evaluate the equation in a different frame and the result must be the same.

Literature:

- S. Weinberg, *The Quantum Theory of Fields. Vol. 1: Foundations.* Cambridge University Press, 1995.
- M. Maggiore, *A Modern Introduction to Quantum Field Theory.* Oxford University Press, 2005.
- W.-K. Tung, *Group Theory in Physics.* World Scientific, 1985.
- F. Scheck, *Quantum Physics.* Springer, 2007.
- B. Thaller, *The Dirac Equation.* Springer, 1992.