3 Dirac field

Earlier in Eqs. (1.31–1.32) we claimed that, under Poincaré transformations $x' = \Lambda x + a$, a generic set of *classical* fields $\Phi_i(x)$ transforms as

$$\Phi_i'(x') = D(\Lambda)_{ij}\, \Phi_j(x)\,, \tag{3.1}$$

and the quantum version of this relation for field *operators* was given in Eq. (2.58):

$$U(\Lambda, a)\, \Phi_i(x)\, U(\Lambda, a)^{-1} = D(\Lambda)^{-1}_{ij}\, \Phi_j(\Lambda x + a)\,. \tag{3.2}$$

We have already worked out the structure of $U(\Lambda, a)$ (at least a little bit): it contains the generators P^μ of translations and $M^{\mu\nu}$ of Lorentz transformations, which are now understood as operators on the Fock space. For example, we established the momentum operator for a free scalar theory in Eq. (2.22), and it is easy to show that it satisfies indeed the Lie algebra relation $[P^\mu, P^\nu] = 0$.

Irreducible representations of the Lorentz group. The missing link in both cases is the matrix $D(\Lambda)$. Because it refers to the indices i and j in the equations above, it classifies which types of fields can actually appear in a Lagrangian: scalar, Dirac, vector fields etc. We will see that $D(\Lambda)$ also provides the **spin** contribution to observables. For scalar fields $D(\Lambda) = 1$ and so we could simply ignore it. In general, $D(\Lambda)$ is a finite-dimensional irreducible representation matrix of the Lorentz group, so it must share the same structure with $U(\Lambda, 0)$:

$$D(\Lambda) = e^{\frac{i}{2}\,\varepsilon_{\mu\nu}\, M^{\mu\nu}} = e^{i\boldsymbol{\phi}\cdot\boldsymbol{J} + i\boldsymbol{s}\cdot\boldsymbol{K}}\,, \qquad M^{ij} = -\varepsilon_{ijk}\, J^k\,, \qquad M^{0i} = K^i\,. \tag{3.3}$$

That is, in an n-dimensional representation $D(\Lambda)$, $M^{\mu\nu}$, $\boldsymbol{J}$ and $\boldsymbol{K}$ are $n \times n$ matrices. Of course $M^{\mu\nu}$ is not the same as the Fock-space operator that was just mentioned before, but let's keep the generic notation for the moment to avoid clutter. What do these matrices look like? Can they have any dimensionality?

Let's build a Lorentz tensor of rank n. It is defined by the transformation law

$$(T')^{\mu\nu\dots\tau} = \underbrace{\Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta \dots \Lambda^\tau{}_\lambda}_{n \text{ times}}\, T^{\alpha\beta\dots\lambda}\,, \tag{3.4}$$

so we can always construct the representation matrices $\Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta \cdots$ of the Lorentz transformation as the outer product $\mathbf{4} \otimes \mathbf{4} \otimes \cdots$ of the 4-dimensional defining representation Λ. However, these representations are not irreducible. Take for example the 4×4 tensor $T^{\mu\nu}$, which has in principle 16 components. Its trace, its antisymmetric component, and its symmetric and traceless part,

$$S = T^\alpha{}_\alpha, \qquad A^{\mu\nu} = \tfrac{1}{2}\,(T^{\mu\nu} - T^{\nu\mu}), \qquad S^{\mu\nu} = \tfrac{1}{2}\,(T^{\mu\nu} + T^{\nu\mu}) - \tfrac{1}{4}\, g^{\mu\nu}\, S, \tag{3.5}$$

do not mix under Lorentz transformations: an (anti-) symmetric tensor is still (anti-) symmetric after the transformation, and the trace S is Lorentz-invariant. The trace is one-dimensional, the antisymmetric part defines a 6-dimensional subspace, and the

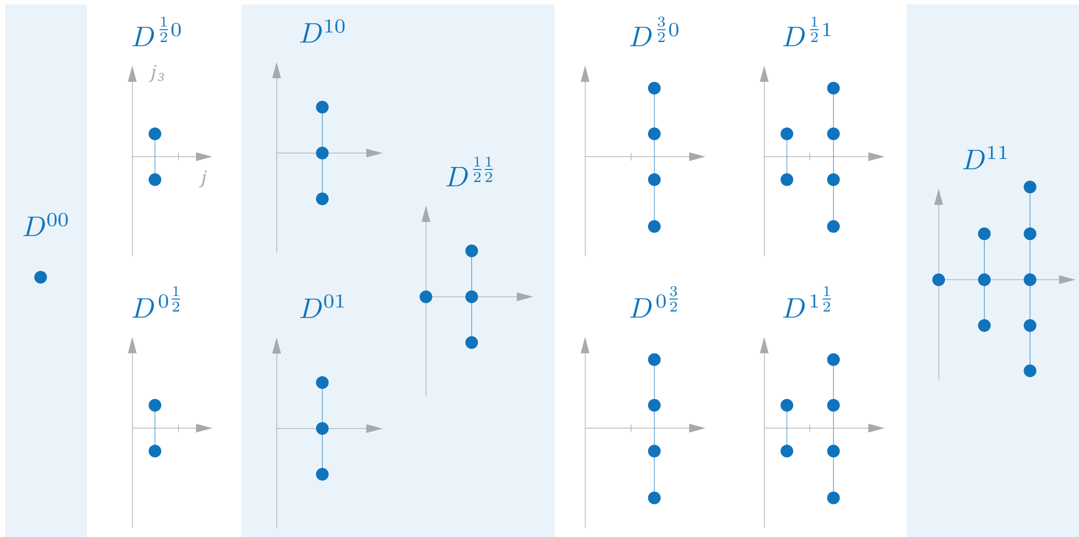

FIGURE 3.1: Multiplets of the Lorentz group: tensor (shaded) vs. spinor representations. The number of states in a multiplet gives the dimension of the representation.

symmetric and traceless part a 9-dimensional subspace. Therefore, we have the decomposition $\mathbf{4}\otimes\mathbf{4}=\mathbf{1}\oplus\mathbf{6}\oplus\mathbf{9}$, which means there must be at least representations with dimensions 1, 4, 6 and 9. How many more are there?

There is a simple way to classify the irreducible representations of the Lorentz group. If we define

$$\boldsymbol{A}=\tfrac{1}{2}\,(\boldsymbol{J}-i\boldsymbol{K}), \qquad \boldsymbol{B}=\tfrac{1}{2}\,(\boldsymbol{J}+i\boldsymbol{K}) \tag{3.6}$$

and calculate their commutator relations using Eq. (2.56), we obtain two copies of an $SU(2)$ algebra with hermitian generators A_i and B_i:

$$[A_i,A_j]=i\varepsilon_{ijk}\,A_k\,, \qquad [B_i,B_j]=i\varepsilon_{ijk}\,B_k\,, \qquad [A_i,B_j]=0\,. \tag{3.7}$$

We are familiar with $SU(2)$: the two Casimir operators $\boldsymbol{A}^2$ and $\boldsymbol{B}^2$ have eigenvalues $a\,(a+1)$ and $b\,(b+1)$, hence there are two quantum numbers $a,\,b=0,\,\frac12,\,1,\dots$ to label the multiplets. We denote the irreducible representations by D^{ab}; their dimension must be $(2a+1)(2b+1)$. The generator of rotations is $\boldsymbol{J}=\boldsymbol{A}+\boldsymbol{B}$, so we can use the $SU(2)$ angular momentum addition rules to construct the states within each multiplet: the states come with all possible spins $j=|a-b|\dots a+b$, where j_3 goes from $-j$ to j. The multiplets are visualized in Fig. 3.1.

The 'tensor representations', where $a+b$ is integer (the shaded multiplets in Fig. 3.1), are the actual irreducible representations of the Lorentz group that can be constructed via Eq. (3.4):

- **Lorentz scalars** transform under the trivial representation D^{00}, where the generator is $M^{\mu\nu}=0$ and the representation matrix is $D(\Lambda)=1$.

- A **Lorentz vector** transforms under the four-dimensional vector representation $D^{\frac{1}{2}\frac{1}{2}}$. It plays a special role because the transformation matrix is $D(\Lambda) = \Lambda$ itself, and it can be used to construct all further (reducible) tensor representations according to Eq. (3.4). The generator $M^{\mu\nu}$ has the form of Eq. (2.54).
- A **symmetric and traceless tensor** $S^{\mu\nu}$ transforms under the 9-dimensional 'tensor' representation D^{11}.
- An **antisymmetric tensor** $A^{\mu\nu}$ transforms under the six-dimensional antisymmetric representation. If $A^{\mu\nu}$ is real it is also irreducible; if it is complex (which it is in Euclidean space) it can be further decomposed into a self-dual (D^{10}) and an anti-self-dual representation (D^{01}), depending on the sign of the condition $A^{\mu\nu} = \pm\frac{i}{2}\,\varepsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}$.

These are the representations **1**, **4**, **6** and **9** that we anticipated above. However, what is more interesting in view of Dirac fields are the **spinor representations** where $a+b$ is half-integer. They are not representations of the Lorentz group but rather of the group $SL(2,C)$, which is the set of complex 2×2 matrices with unit determinant. Like the Lorentz group, it also depends on six real parameters and it has the same Lie algebra. From the point of view of the Lorentz group, the spinor representations are merely *projective* representations, where instead of $D(\Lambda')\,D(\Lambda) = D(\Lambda'\Lambda)$ one has $D(\Lambda')\,D(\Lambda) = \pm D(\Lambda'\Lambda)$, so they are double-valued. However, both of them are physically equivalent and therefore the representations in Fig. 3.1 are all relevant.

The origin of this behavior is the rotational subgroup $SO(3)$ of the Lorentz group which is not simply connected. The projective representations of a group correspond to the representations of its universal covering group: it has the same Lie algebra, which reflects the property of the group close to the identity, but it is simply connected. In the same way as $SU(2)$ is the double cover of $SO(3)$, the double cover of $SO(3,1)^\uparrow$ is the group $SL(2,\mathbb{C})$. A double-valued projective representation of $SO(3,1)^\uparrow$ corresponds to a single-valued representation of $SL(2,\mathbb{C})$. Similarly, the double cover of the *Euclidean* Lorentz group $SO(4)$ is $SU(2)\times SU(2)$; these are the representations that we actually derived in Fig. 3.1.

The fundamental spinor representations are $D^{\frac{1}{2}0}$ and $D^{0\frac{1}{2}}$ because all other ones can be built from them. They have both dimension two and carry spin $j = 1/2$. Because one of the Casimir eigenvalues a or b is zero, we say that they have definite *chirality*: $D^{\frac{1}{2}0}$ is the left-handed and $D^{0\frac{1}{2}}$ the right-handed representation. We can immediately write down 2×2 matrices that satisfy the $SU(2)$ Lie algebra, namely the Pauli matrices:

$$\left[\frac{\sigma^i}{2},\frac{\sigma^j}{2}\right] = i\varepsilon_{ijk}\,\frac{\sigma^k}{2}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.8}$$

Therefore the generators and transformation matrices are

$$\begin{array}{lll} D^{\frac{1}{2}0}: \quad \boldsymbol{A} = \frac{\boldsymbol{\sigma}}{2}\,,\ \boldsymbol{B} = 0 & & \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}\,,\ \boldsymbol{K} = i\frac{\boldsymbol{\sigma}}{2} & & D_L(\Lambda) = e^{i\boldsymbol{\phi}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{s}\cdot\frac{\boldsymbol{\sigma}}{2}}\,, \\ & \Rightarrow & & \Rightarrow & \\ D^{0\frac{1}{2}}: \quad \boldsymbol{A} = 0 \quad \boldsymbol{B} = \frac{\boldsymbol{\sigma}}{2} & & \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}\,,\ \boldsymbol{K} = -i\frac{\boldsymbol{\sigma}}{2} & & D_R(\Lambda) = e^{i\boldsymbol{\phi}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{s}\cdot\frac{\boldsymbol{\sigma}}{2}}\,. \end{array} \tag{3.9}$$

The representation matrices $D_{L,R}(\Lambda) \in SL(2,\mathbb{C})$ are complex 2×2 matrices, and the corresponding spinors are left- and right-handed **Weyl spinors** ψ_L, ψ_R that transform as

$$\psi_L'(x') = D_L(\Lambda)\,\psi_L(x)\,, \qquad \psi_R'(x') = D_R(\Lambda)\,\psi_R(x)\,. \tag{3.10}$$

We can check that these are only projective representations. Consider, for example, a rotation by $\phi = 2\pi$ around the $z-$axis: the Lorentz transformation is $\Lambda = 1$, but the representation matrices become $D_{L,R}(1) = e^{i\pi\sigma_3} = \cos\pi + i\sigma_3 \sin\pi = -1$, and only a rotation by 4π will bring them back to 1.

In principle, the Weyl representation would be sufficient to describe spin-$\frac{1}{2}$ fields. However, the problem is that under a parity transformation the rotation generators are invariant whereas the boost generators change their sign: $\boldsymbol{J} \to \boldsymbol{J}$, $\boldsymbol{K} \to -\boldsymbol{K}$. Therefore, parity exchanges $\boldsymbol{A} \leftrightarrow \boldsymbol{B}$ in Eq. (3.6) and transforms the two fundamental representations into each other. A theory that is invariant under parity (such as QED and QCD, but not the weak interaction) must necessarily include both doublets, because we cannot write down a parity-invariant Lagrangian with ψ_L or ψ_R alone. In such a combined Lagrangian the dynamics will couple ψ_L and ψ_R together. This is a consequence of Eq. (3.9) because $D_L(\Lambda)^\dagger = D_R(\Lambda)^{-1}$, and a Lorentz-invariant Lagrangian will contain terms $\sim \psi_L^\dagger \psi_R$, $\psi_R^\dagger \psi_L$ that are separately Lorentz-invariant.

Instead of carrying around the left- and right-handed Weyl spinors, it is more convenient to combine them into **Dirac spinors** ψ_α with $\alpha = 1 \dots 4$. They can be constructed as the direct sums of ψ_L and ψ_R, hence we denote the (reducible) Dirac representation by $D^{\frac{1}{2}0} \oplus D^{0\frac{1}{2}}$:

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{\sigma}/2 & 0 \\ 0 & \boldsymbol{\sigma}/2 \end{pmatrix} =: \frac{\boldsymbol{\Sigma}}{2}, \qquad \boldsymbol{K} = \begin{pmatrix} i\boldsymbol{\sigma}/2 & 0 \\ 0 & -i\boldsymbol{\sigma}/2 \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{3.11}$$

The resulting generator $M^{\mu\nu}$ constructed via Eq. (3.3) is consequently a 4×4 matrix that satisfies again the Lorentz algebra relation (2.49). It leads to a four-dimensional transformation matrix

$$D(\Lambda) = e^{\frac{i}{2}\varepsilon_{\mu\nu} M^{\mu\nu}} = e^{i\boldsymbol{\phi}\cdot\boldsymbol{J} + i\boldsymbol{s}\cdot\boldsymbol{K}} \tag{3.12}$$

which transforms the spinors as $\psi'_\alpha(x') = D(\Lambda)_{\alpha\beta}\,\psi_\beta(x)$.

Clifford algebra. It is still desirable to have a manifestly covariant notation. This is where the Clifford algebra comes in: it is the algebra spanned by the $n \times n$ matrices γ^μ, with $\mu = 0 \dots 3$, so that the anticommutator is

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbb{1}_{n\times n}\,. \tag{3.13}$$

This implies $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$ and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$. The Clifford algebra is quite useful because every representation of it *induces* a representation of the Lorentz algebra via the definition

$$M^{\mu\nu} := -\frac{i}{4}\,[\gamma^\mu, \gamma^\nu]\,. \tag{3.14}$$

(Ex) That is, by using the anticommutator relation (3.13) one can show that $M^{\mu\nu}$ satisfies the Lorentz algebra relation (2.49). Consequently, for $n = 4$ there must be an explicit form for the $\gamma-$matrices where $M^{\mu\nu}$ reproduces Eq. (3.11); it is called the *chiral* or *Dirac* representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \Leftrightarrow \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{3.15}$$

where we abbreviated $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$. We also define

$$\gamma^5 = \gamma_5 := \frac{i}{4!}\,\varepsilon_{\mu\nu\rho\sigma}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3 \overset{\text{chiral rep.}}{=} \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{3.16}$$

with the properties $(\gamma_5)^\dagger = \gamma_5$, $(\gamma_5)^2 = 1$ and $\{\gamma^\mu, \gamma_5\} = 0$. The totally antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ is defined as

$$\varepsilon_{\mu\nu\rho\sigma} = \left\{ \begin{array}{cl} +1 & \text{if } \; \mu\nu\rho\sigma \; \text{ is an even permutation of } \; 0123 \\ -1 & \text{if } \; \mu\nu\rho\sigma \; \text{ is an odd permutation of } \; 0123 \\ 0 & \text{otherwise.} \end{array} \right\} \tag{3.17}$$

It switches sign if *spatial* indices are raised or lowered, which entails $\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma}$. The matrix γ_5 is useful for constructing the *chiral projectors* $(1 \pm \gamma_5)/2$ onto the Weyl spinors:

$$\frac{1-\gamma_5}{2}\,\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \qquad \frac{1+\gamma_5}{2}\,\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}. \tag{3.18}$$

The chiral representation is where the group structure is most transparent because the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ are the direct sums of the two-dimensional matrices. Expressed in terms of gamma matrices they are given by $\boldsymbol{\Sigma} = \gamma_5\gamma^0\boldsymbol{\gamma}$ and $\boldsymbol{K} = -\frac{i}{2}\gamma^0\boldsymbol{\gamma}$, which follows from Eqs. (3.3), (3.14) and (3.16). It is also practical for calculations in the ultrarelativistic limit where masses can be neglected.

It follows from Eq. (3.13) that with every invertible matrix U also $U\gamma^\mu U^{-1}$ is a representation of the Clifford algebra, and $U\psi$ is the spinor in the new representation. For example, the *standard* representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.19}$$

is frequently used because it is convenient for calculations in the non-relativistic limit. It emerges from the chiral representation through the matrix

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \quad \Rightarrow \quad \psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} =: \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \tag{3.20}$$

By multiplying the $\gamma-$matrices with each other one can form a complete system of 4×4 matrices, which consists of 16 matrices $\Gamma_1 \dots \Gamma_{16}$:

$$\mathbb{1}, \qquad \gamma^\mu, \qquad \sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu], \qquad \gamma^\mu\gamma^5, \qquad \gamma^5\,. \tag{3.21}$$

They are orthonormal with respect to the scalar product $\frac{1}{4}\,\mathrm{Tr}\,(\Gamma_i^\dagger\,\Gamma_j) = \delta_{ij}$ and (except for Γ_1) traceless: $\mathrm{Tr}\,\Gamma_i = \delta_{i1}$. Therefore we can express any 4×4 matrix by

$$A = \sum_{i=1}^{16} c_i\,\Gamma_i\,, \qquad c_i = \frac{1}{4}\,\mathrm{Tr}\,(\Gamma_i^\dagger\,A)\,. \tag{3.22}$$

Lorentz bilinears. How can we construct Lorentz invariants from a spinor $\psi(x)$? We already know that under Lorentz transformations we have $\psi'(x') = D(\Lambda)\psi(x)$. Let's try the combination

$$\psi^\dagger(x)\,\psi(x) \quad \to \quad \psi'^\dagger(x')\,\psi'(x') = \psi^\dagger(x)\,D(\Lambda)^\dagger D(\Lambda)\,\psi(x) \tag{3.23}$$

For $D(\Lambda)^\dagger = D(\Lambda)^{-1}$ this would be a Lorentz scalar. However, $D(\Lambda)$ cannot be unitary because it contains the boosts: $\boldsymbol{J}$ is hermitian but $\boldsymbol{K}$ is antihermitian, and consequently $M^{\mu\nu}$ cannot be hermitian:

$$M^\dagger_{\mu\nu} = \left(-\frac{i}{4}\,[\gamma^\mu,\gamma^\nu]\right)^\dagger = -\frac{i}{4}\,[\gamma^{\mu\dagger},\gamma^{\nu\dagger}] \neq -\frac{i}{4}\,[\gamma^\mu,\gamma^\nu]\,. \tag{3.24}$$

From the point of view of the Clifford algebra, it is impossible to make all $\gamma-$matrices hermitian: since $(\gamma^0)^2 = 1$, γ^0 has real eigenvalues, but $(\gamma^i)^2 = -1$ and therefore the eigenvalues of γ^i are imaginary. What we can write instead is

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i \quad \Rightarrow \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{3.25}$$

and therefore also $\gamma^0 M^\dagger_{\mu\nu}\gamma^0 = M_{\mu\nu}$ and $\gamma^0 D(\Lambda)^\dagger\gamma^0 = D(\Lambda)^{-1}$. This is why we define the **conjugate spinor**

$$\overline{\psi} := \psi^\dagger\gamma^0 \quad \Rightarrow \quad \overline{\psi}'(x') = \psi^\dagger(x)\,D(\Lambda)^\dagger\gamma^0 = \psi^\dagger(x)\,\gamma^0 D(\Lambda)^{-1} = \overline{\psi}(x)\,D(\Lambda)^{-1}\,, \tag{3.26}$$

because it makes the quantity $\overline{\psi}\psi$ invariant:

$$\overline{\psi}'(x')\,\psi'(x') = \overline{\psi}(x)\,D(\Lambda)^{-1}D(\Lambda)\,\psi(x) = \overline{\psi}(x)\,\psi(x)\,. \tag{3.27}$$

(Ex) Similarly, one can use the identity $D(\Lambda)^{-1}\,\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$ to show that $\overline{\psi}\gamma^\mu\psi$ transforms like a Lorentz vector:

$$\overline{\psi}'(x')\,\gamma^\mu\psi'(x') = \overline{\psi}(x)\,D(\Lambda)^{-1}\gamma^\mu D(\Lambda)\,\psi(x) = \Lambda^\mu{}_\nu\,\overline{\psi}(x)\,\gamma^\nu\psi(x)\,. \tag{3.28}$$

Moreover, when we contract a Lorentz vector with another one, we get a Lorentz scalar:

$$\begin{aligned}\overline{\psi}'(x')\,\gamma^\mu\,\partial'_\mu\,\psi'(x') &= \overline{\psi}(x)D(\Lambda)^{-1}\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\,\partial_\nu\,D(\Lambda)\,\psi(x)\\ &= (\Lambda^{-1})^\nu{}_\mu\,\Lambda^\mu{}_\rho\,\overline{\psi}(x)\,\gamma^\rho\,\partial_\nu\,\psi(x)\\ &= \overline{\psi}(x)\,\gamma^\nu\partial_\nu\,\psi(x)\,.\end{aligned} \tag{3.29}$$

From now on we will use the Feynman slash notation $\not{A} = \gamma^\mu A_\mu$ for a generic four-vector A_μ, so the last expression simply becomes $\overline{\psi}\,\not{\partial}\,\psi$. The definition also entails $\not{A}^2 = A^2$. Note that only the combinations $\overline{\psi}\,\not{A}\,\psi$ are Lorentz-invariant but not $\not{A}$ itself. (Also, be careful with derivatives because $\not{A} = \gamma^\mu A_\mu = \gamma^0 A^0 - \boldsymbol{\gamma}\cdot\boldsymbol{A}$ whereas $\not{\partial} = \gamma^\mu\partial_\mu = \gamma^0\partial^0 + \boldsymbol{\gamma}\cdot\boldsymbol{\nabla}$.) Finally, one can show that the bilinears

$$\overline{\psi}\,i\gamma^5\psi, \qquad \overline{\psi}\gamma^\mu\gamma^5\psi, \qquad \overline{\psi}\sigma^{\mu\nu}\psi \tag{3.30}$$

transform like a pseudoscalar, axialvector and tensor, respectively. We will discuss this later in the context of discrete symmetries.

Dirac Lagrangian. The simplest Lorentz scalars that we can build from $\psi(x)$ and $\overline{\psi}(x)$ and include non-trivial dynamics are $\overline{\psi}\,\psi$ and $\overline{\psi}\,\partial\!\!\!/\,\psi$. Unlike in the scalar case, we can construct a Lorentz-invariant action already with those two terms alone (which contain only one derivative):

$$S = \int d^4x\,\mathcal{L} = \int d^4x\,\overline{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x)\,. \tag{3.31}$$

The factor i is necessary to make $\mathcal{L}$ real, the dimension of the field is $[\psi] = 3/2$, and m is a mass. Since $\psi(x)$ is a complex field we treat ψ and $\psi^\dagger$ (or equivalently $\overline{\psi}$) as independent when deriving the Euler-Lagrange equations of motion:

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} = (i\partial\!\!\!/ - m)\psi\,, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0 \qquad \Rightarrow \qquad (i\partial\!\!\!/ - m)\,\psi = 0\,. \tag{3.32}$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\overline{\psi}\,, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\overline{\psi}\gamma^\mu \qquad \Rightarrow \qquad \overline{\psi}\,(i\overleftarrow{\partial\!\!\!/} + m) = 0\,, \tag{3.33}$$

where $\overleftarrow{\partial}_\mu$ means that the derivative acts to the *left* instead of the right. When we take the resulting **Dirac equation** $(i\partial\!\!\!/ - m)\,\psi = 0$ and apply $(i\partial\!\!\!/ + m)$ from the left, we obtain Klein-Gordon equations for each component of the Dirac field:

$$(i\partial\!\!\!/ + m)\,(i\partial\!\!\!/ - m)\,\psi = -(\Box + m^2)\,\psi = 0\,. \tag{3.34}$$

However, since the Dirac equation is a first-order equation it provides a *stronger* constraint on $\psi(x)$ than the KG equation, which is of second order.

Symmetries and currents. We can adapt the discussion of the Noether theorem to spinor fields without any modifications. The Noether current of Eq. (1.40) takes the form

$$-\,\delta j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi}_\alpha)}\,\delta\overline{\psi}_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\,\delta\psi_\alpha - T^{\mu\nu}\delta x_\nu = i\overline{\psi}\,\gamma^\mu\delta\psi - T^{\mu\nu}\delta x_\nu\,, \tag{3.35}$$

with the energy-momentum tensor of the Dirac field given by

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi}_\alpha)}\,\partial^\nu\overline{\psi}_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\,\partial^\nu\psi_\alpha - g^{\mu\nu}\mathcal{L} = i\overline{\psi}\,\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\mathcal{L}\,. \tag{3.36}$$

- Let's start with **translation invariance**: each component of the Dirac field behaves like a scalar under translations, $\psi'_\alpha(x+a) = \psi_\alpha(x) \Leftrightarrow \delta\psi_\alpha = 0$ and $\delta x^\mu = a^\mu$, and therefore the conserved current is the energy-momentum tensor itself: $\partial_\mu T^{\mu\nu} = 0$. This can be easily checked: $\mathcal{L} = 0$ for solutions of the Dirac equation, so the last term in Eq. (3.36) vanishes, and the derivative of the first term also becomes zero when the Dirac equations are inserted. The conserved charges are the Hamiltonian of the Dirac field and its total momentum:

$$\begin{aligned} H &= \int d^3x\,T^{00} = \int d^3x\,\overline{\psi}\,(i\gamma^0\partial^0 - i\partial\!\!\!/ + m)\,\psi = \int d^3x\,\overline{\psi}\,(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi\,, \\ P^k &= \int d^3x\,T^{0k} = \int d^3x\,\psi^\dagger\,i\partial^k\,\psi = \int d^3x\,\psi^\dagger(-i\nabla_k)\,\psi\,. \end{aligned} \tag{3.37}$$

- The implications of **Lorentz invariance** can be worked out in a similar fashion. Lorentz transformations have the form

$$\begin{aligned} x' &= \Lambda x \\ \psi'(\Lambda x) &= D(\Lambda)\,\psi(x) \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} \delta x^\mu &= \varepsilon^{\mu\nu} x_\nu \\ \delta\psi &= i\varepsilon_{\mu\nu}\, M^{\mu\nu}\psi \end{aligned} \tag{3.38}$$

and the infinitesimal current becomes

$$\begin{aligned} -\delta j^\mu &= -\tfrac{1}{2}\,\varepsilon_{\alpha\beta}\,\overline{\psi}\,\gamma^\mu M^{\alpha\beta}\psi - T^{\mu\alpha}\varepsilon_{\alpha\beta}x^\beta \\ &= -\tfrac{1}{2}\,\varepsilon_{\alpha\beta}\,\underbrace{\left(\overline{\psi}\,\gamma^\mu M^{\alpha\beta}\psi + T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha\right)}_{=:\, m^{\mu,\alpha\beta}}\,. \end{aligned} \tag{3.39}$$

The angular momentum density $m^{\mu,\alpha\beta}$ is the analogue of Eq. (1.44) from the scalar case and it is conserved: $\partial_\mu\, m^{\mu,\alpha\beta} = 0$. However, now it contains an additional spin contribution. Using the definition of $L^{\alpha\beta}$ in Eq. (1.44), we write

$$T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha = \overline{\psi}\,\gamma^\mu L^{\alpha\beta}\psi + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha})\mathcal{L}\,, \tag{3.40}$$

which allows us to combine the spin part $M^{\mu\nu} = -\frac{i}{4}[\gamma^\mu,\gamma^\nu]$ with the orbital part $L^{\mu\nu}$ into a total angular momentum tensor $J^{\mu\nu} = L^{\mu\nu} + M^{\mu\nu}$:

$$m^{\mu,\alpha\beta} = \overline{\psi}\,\gamma^\mu J^{\alpha\beta}\psi + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha})\mathcal{L}\,. \tag{3.41}$$

Consider for example the invariance under rotations: the corresponding generator for $M^{\mu\nu}$ is $\boldsymbol{\Sigma}/2$, and its analogue for $L^{\mu\nu}$ is the three-vector $\boldsymbol{L} = \boldsymbol{x}\times(-i\boldsymbol{\nabla})$. Hence, the quantity that is conserved under rotations is the total angular momentum $\widetilde{\boldsymbol{J}}$ of the field:

$$\int d^3x\, m^{0,ij} = \int d^3x\, \psi^\dagger J^{ij}\psi =: -\varepsilon_{ijk}\,\widetilde{J}^k\,, \qquad \widetilde{\boldsymbol{J}} = \int d^3x\,\psi^\dagger\left(\boldsymbol{L} + \frac{\boldsymbol{\Sigma}}{2}\right)\psi\,. \tag{3.42}$$

- An example for **internal symmetries** is the $U(1)$ transformation

$$\psi' = e^{i\varepsilon}\,\psi\,, \quad \overline{\psi}' = e^{-i\varepsilon}\,\overline{\psi} \qquad \Rightarrow \qquad \delta\psi = i\varepsilon\,\psi\,, \quad \delta\overline{\psi} = -i\varepsilon\,\overline{\psi}\,, \tag{3.43}$$

with $\varepsilon \in \mathbb{R}$ constant, which leaves the Dirac Lagrangian invariant. It leads to the conserved **vector current** and charge

$$j^\mu_V = \overline{\psi}\,\gamma^\mu\psi\,, \qquad Q_V = \int d^3x\,\psi^\dagger\,\psi\,. \tag{3.44}$$

- Another less obvious symmetry is the **axial** $U(1)_A$ symmetry

$$\psi' = e^{i\varepsilon\,\gamma_5}\,\psi\,, \qquad \overline{\psi}' = {\psi'}^\dagger\gamma^0 = \psi^\dagger\, e^{-i\varepsilon\,\gamma_5}\,\gamma^0 = \psi^\dagger\gamma^0\, e^{+i\varepsilon\,\gamma_5} = \overline{\psi}\, e^{i\varepsilon\,\gamma_5}\,, \tag{3.45}$$

which is only realized in the massless limit ($m = 0$) because the Lagrangian transforms as

$$\overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi \quad \rightarrow \quad \overline{\psi}\, e^{i\varepsilon\,\gamma_5}(i\partial\!\!\!/ - m)\, e^{i\varepsilon\,\gamma_5}\,\psi = \overline{\psi}\,(i\partial\!\!\!/ - m\, e^{2i\varepsilon\,\gamma_5})\,\psi\,. \tag{3.46}$$

In these rearrangements the relation $e^{i\varepsilon\gamma_5} = \cos\varepsilon + i\gamma_5 \sin\varepsilon$ is helpful, which holds because $(\gamma_5)^2 = \mathbb{1}$, and we also used $\gamma_5 = \gamma_5^\dagger$ and $\gamma_5\,\gamma^\mu = -\gamma^\mu\gamma_5$. The corresponding axialvector current is

$$j_A^\mu = \overline{\psi}\,\gamma^\mu\gamma_5\,\psi \tag{3.47}$$

and we can check explicitly that it is only conserved for $m = 0$:

$$\partial_\mu\, j_A^\mu = \overline{\psi}\,\overleftarrow{\not\partial}\,\gamma_5\,\psi - \overline{\psi}\,\gamma_5\,\not\partial\,\psi = 2im\,\overline{\psi}\,\gamma_5\,\psi\,. \tag{3.48}$$

This identity goes by the name **PCAC relation** (partially conserved axialvector current). Its underlying origin is that the left- and right-handed fields ψ_L, ψ_R decouple for $m = 0$, which leads to an enlarged *chiral symmetry* of the Lagrangian (see discussion below). Chiral symmetry has a rather prominent status in QCD: in a theory with N fermion flavors, the massless Lagrangian is invariant under $U(1)_V \times SU(N)_V \times SU(N)_A \times U(1)_A$. The latter two are explicitly broken by the quark masses, but $SU(N)_A$ is also *spontaneously* broken (which entails that the pions are Goldstone bosons), whereas $U(1)_A$ is *anomalously* broken at the quantum level.

Massless fields. Let's rewrite the Dirac Lagrangian (3.31) in terms of Weyl spinors. From Eq. (3.15) we have

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \overline{\psi} = \psi^\dagger\gamma^0 = (\psi_R^\dagger, \psi_L^\dagger)\,, \qquad \not\partial = \begin{pmatrix} 0 & \sigma\cdot\partial \\ \overline{\sigma}\cdot\partial & 0 \end{pmatrix} \tag{3.49}$$

and the Dirac Lagrangian becomes

$$\mathcal{L} = i\psi_R^\dagger\,\sigma\cdot\partial\,\psi_R + i\psi_L^\dagger\,\overline{\sigma}\cdot\partial\,\psi_L - m\,(\psi_R^\dagger\,\psi_L + \psi_L^\dagger\,\psi_R)\,. \tag{3.50}$$

If $m = 0$, the left- and right-handed spinors decouple and describe independent degrees of freedom, which is why the limit $m = 0$ is also called **chiral limit**. The corresponding Euler-Lagrange equations are the **Weyl equations:** $i\sigma\cdot\partial\,\psi_R = 0$, $i\overline{\sigma}\cdot\partial\,\psi_L = 0$. With the ansatz $\psi_{R,L}(x) = u_{R,L}(p)\,e^{-ipx}$ they become

$$\begin{aligned} (p\cdot\sigma)\,u_R &= 0 \\ (p\cdot\overline{\sigma})\,u_L &= 0 \end{aligned} \quad\Rightarrow\quad h\,u_{R,L} = \pm\frac{1}{2}\,u_{R,L} \qquad \text{with} \qquad h = \frac{\boldsymbol{\sigma}}{2}\cdot\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\,, \tag{3.51}$$

where h is the **helicity** (the projection of the spin in the momentum direction). Hence, in the limit $m = 0$ the right- and left-handed Weyl spinors are eigenstates of the helicity with eigenvalues $\pm\frac{1}{2}$. If $m \neq 0$, they no longer decouple and it is impossible to define a Lorentz-invariant notion of helicity: in that case particles travel with velocity $v < c$ and it is always possible to find a Lorentz frame where the particle moves in the opposite direction, which causes a change in the helicity.

In the chiral limit $m = 0$ the helicity is Lorentz-invariant (and actually even Poincaré-invariant). That is, in principle we could interpret the two helicity states

ψ_L and ψ_R (whose dynamics also decouple) as two different species of particles. However, parity still transforms them into each other and hence we need both to have a parity-invariant Lagrangian. For this reason we do not identify them as independent degrees of freedom but rather as two polarization states of the same particle. The exception are theories that *break* parity invariance, because in that case it is not necessary to have both chiralities. For example, the right-handed neutrinos do not participate in the weak interaction, and neutrinos in the Standard Model are therefore described by massless left-handed Weyl fields with Lagrangian $i\psi_L^\dagger\,\bar\sigma\cdot\partial\,\psi_L$.

A special case are **Majorana spinors** where ψ_L and ψ_R are not independent but $\psi_R = i\sigma^2\psi_L^*$. The corresponding four-spinor $\psi = (\psi_L, \psi_R)$ is invariant under charge conjugation. This is the spinor analogue of the real scalar field (the condition $\psi = \psi^*$ alone is not Lorentz-invariant because $D(\Lambda)$ is not real), so the corresponding particle would be its own antiparticle. Since in that case we lose the $U(1)$ symmetry, Majorana fields cannot describe fermions that carry a $U(1)$ charge (electric charge, lepton number, etc.). Possible candidates are, again, the neutrinos whose masses are very small but most likely nonzero. If they were Majorana particles, lepton number symmetry would be violated, and experiments on neutrino-less double beta decay aim at detecting such violations.

In general it is quite useful to study massless Dirac particles because scattering matrices often simplify greatly if the particles can be approximated as massless. This is usually realized in QED processes because the electron mass is much smaller compared to other relevant scales. It is also useful in QCD where the light up and down quarks can be treated as nearly massless particles. An interesting feature in the chiral limit is that both fields ψ_L, ψ_R transform now independently under $U(1)$ transformations, which leave the Lagrangian separately invariant:

$$\psi_L' = e^{i\varepsilon_L}\psi_L\,, \qquad \psi_R' = e^{i\varepsilon_R}\psi_R\,. \tag{3.52}$$

The corresponding $U(1)_L\times U(1)_R$ symmetry is called **chiral symmetry**. It is equivalent to the $U(1)_V\times U(1)_A$ symmetry that we encountered above because the conserved left- and right-handed currents are linear combinations of the vector and axialvector currents j_V^μ and j_A^μ. **(Ex)**

Classical solutions of the Dirac equation. Like in the scalar case, the general solutions of the free Dirac equations can be expressed by plane waves with positive- and negative frequency modes:

$$\begin{aligned} \psi_+(x) &= u(\boldsymbol{p})\,e^{-ipx} \\ \psi_-(x) &= v(\boldsymbol{p})\,e^{ipx} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} (\not p - m)\,u(\boldsymbol{p}) &= 0\,, \\ (\not p + m)\,v(\boldsymbol{p}) &= 0\,. \end{aligned} \tag{3.53}$$

We recover $p^2 = m^2$ by multiplying the equations with $\not p \pm m$, so these are indeed solutions of the Dirac equation. We have again chosen $p^0 = +E_p = +\sqrt{\boldsymbol{p}^2 + m^2}$ to be positive and put the sign instead in the exponential; we could have also started with e^{-ipx} alone and distinguish the two solutions by $p^0 = \pm E_p$ (with a change $\boldsymbol{p}\to -\boldsymbol{p}$). The Dirac equation can be written in the form

$$\begin{aligned} (p^0\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} - m)\,u(\boldsymbol{p}) &= 0 \\ (p^0\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m)\,v(\boldsymbol{p}) &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \gamma^0(\boldsymbol{p}\cdot\boldsymbol{\gamma} + m)\,u(\boldsymbol{p}) &= E_p\,u(\boldsymbol{p})\,, \\ \gamma^0(\boldsymbol{p}\cdot\boldsymbol{\gamma} + m)\,v(-\boldsymbol{p}) &= -E_p\,v(-\boldsymbol{p})\,. \end{aligned} \tag{3.54}$$

so that $u(\boldsymbol{p})$ and $v(-\boldsymbol{p})$ are eigenstates of the Dirac Hamiltonian with eigenvalues $\pm E_p$.

In the chiral representation we can write $u = (u_L, u_R)$, and with the explicit form of the $\gamma-$matrices in Eq. (3.15) the Dirac equation becomes

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar\sigma & -m \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \qquad \Rightarrow \qquad \begin{aligned} (p\cdot\sigma)\,u_R &= m\,u_L\,, \\ (p\cdot\bar\sigma)\,u_L &= m\,u_R\,. \end{aligned} \tag{3.55}$$

These two equations are consistent because $(p\cdot\sigma)(p\cdot\bar\sigma) = p^2 = m^2$:

$$(p\cdot\sigma)(p\cdot\bar\sigma) = p_0^2 - p^i\,p^j\,\sigma^i\,\sigma^j = p_0^2 - \boldsymbol{p}^2 = p^2 = m^2\,. \tag{3.56}$$

(Use $\{\sigma^i, \sigma^j\} = 2\delta_{ij}$). Note that ψ_L and ψ_R are no longer helicity eigenstates because of the mass term. Instead, their solution can be written as

$$u_L = \sqrt{p\cdot\sigma}\,\xi\,, \qquad u_R = \sqrt{p\cdot\bar\sigma}\,\xi\,, \tag{3.57}$$

where ξ_s with $s = \pm 1$ are two-component spinors that we normalize to $\xi_s^\dagger\,\xi_{s'} = \delta_{ss'}$. The analogous analysis for negative-frequency modes gives

$$v_L = \sqrt{p\cdot\sigma}\,\eta\,, \qquad v_R = -\sqrt{p\cdot\bar\sigma}\,\eta\,, \tag{3.58}$$

so that we obtain in total

$$u_s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix}, \qquad v_s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar\sigma}\,\eta_s \end{pmatrix}. \tag{3.59}$$

The two components of s can be interpreted as the spin direction. For example, if we choose the basis for the two-component spinors ξ_s as

$$\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.60}$$

they are eigenvectors of the spin matrix $\sigma^3/2$ with eigenvalues $\pm\frac{1}{2}$, so they describe spinors with spin $\pm\frac{1}{2}$ in $z-$direction.

Using the explicit form of the spinors, it is easy to prove the orthogonality relations **(Ex)**

$$\begin{aligned} \bar u_s(\boldsymbol{p})\,u_{s'}(\boldsymbol{p}) &= 2m\,\delta_{ss'}\,, & \bar u_s(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) &= 0\,, & u_s^\dagger(\boldsymbol{p})\,u_{s'}(\boldsymbol{p}) &= 2E_p\,\delta_{ss'}\,, \\ \bar v_s(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) &= -2m\,\delta_{ss'}\,, & \bar v_s(\boldsymbol{p})\,u_{s'}(\boldsymbol{p}) &= 0\,, & v_s^\dagger(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) &= 2E_p\,\delta_{ss'}\,, \end{aligned} \tag{3.61}$$

as well as the completeness relations

$$\sum_s u_s(\boldsymbol{p})\,\bar u_s(\boldsymbol{p}) = \not{p} + m\,, \qquad \sum_s v_s(\boldsymbol{p})\,\bar v_s(\boldsymbol{p}) = \not{p} - m\,. \tag{3.62}$$

Be careful because $u_s^\dagger(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) \neq 0$ and $v_s^\dagger(\boldsymbol{p})\,u_{s'}(\boldsymbol{p}) \neq 0$, but instead one has

$$u_s^\dagger(\boldsymbol{p})\,v_{s'}(-\boldsymbol{p}) = v_s^\dagger(\boldsymbol{p})\,u_{s'}(-\boldsymbol{p}) = 0\,. \tag{3.63}$$

The general solutions of the Dirac equation can be written as

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E_p}\sum_s\left(a_s(\boldsymbol{p})\,u_s(\boldsymbol{p})\,e^{-ipx} + b_s^*(\boldsymbol{p})\,v_s(\boldsymbol{p})\,e^{ipx}\right)_{p^0=E_p}\,. \tag{3.64}$$

If we define the positive- and negative-energy projectors **(Ex)**

$$\Lambda_\pm(\boldsymbol{p}) = \frac{1}{2}\left(\mathbb{1} \pm \frac{\not{p}}{m}\right) = \frac{\pm\not{p}+m}{2m}\,, \quad \Rightarrow \quad \begin{array}{l}\Lambda_\pm(\boldsymbol{p})^2 = \Lambda_\pm(\boldsymbol{p}),\\ \Lambda_\pm(\boldsymbol{p})\,\Lambda_\mp(\boldsymbol{p}) = 0\end{array} \tag{3.65}$$

and write $w_s^+(\boldsymbol{p}) = u_s(\boldsymbol{p})$ and $w_s^-(\boldsymbol{p}) = v_s(\boldsymbol{p})$, then the Dirac equation simply becomes

$$\Lambda_\mp(\boldsymbol{p})\,w_s^\pm(\boldsymbol{p}) = 0\,, \qquad \Lambda_\pm(\boldsymbol{p})\,w_s^\pm(\boldsymbol{p}) = w_s^\pm(\boldsymbol{p}) \tag{3.66}$$

and Eqs. (3.61–3.62) take the compact form

$$\overline{w}_s^\pm(\boldsymbol{p})\,w_{s'}^\pm(\boldsymbol{p}) = \pm 2m\,\delta_{ss'}\,, \qquad \sum_s w_s^\pm(\boldsymbol{p})\,\overline{w}_s^\pm(\boldsymbol{p}) = 2m\,\Lambda_\pm(\boldsymbol{p})\,. \tag{3.67}$$

We can derive more useful relations by adding and subtracting the Dirac equations for $w^\pm$ and $\overline{w}^\pm$:

$$\begin{array}{rl}(\not{p}\mp m)\,w^\pm &= 0\\ \overline{w}^\pm(\not{p}\mp m) &= 0\end{array} \quad\Rightarrow\quad \begin{array}{rl}\overline{w}^\pm\,\mathcal{O}\,(\not{p}\mp m)\,w^\pm &= 0\\ \overline{w}^\pm(\not{p}\mp m)\,\mathcal{O}\,w^\pm &= 0\end{array} \quad\Rightarrow\quad \begin{array}{rl}\overline{w}^\pm\{\mathcal{O},\not{p}\}w^\pm &= \pm 2m\,\overline{w}^\pm\mathcal{O}w^\pm\,,\\ \overline{w}^\pm[\mathcal{O},\not{p}]w^\pm &= 0\,,\end{array} \tag{3.68}$$

where $\mathcal{O}$ is some combination of Dirac matrices. For example, it follows that

$$\overline{w}^\pm\gamma_5\,w^\pm = 0\,, \qquad \overline{w}^\pm\gamma^\mu w^\pm = 2p^\mu\,, \quad \text{etc.} \tag{3.69}$$

In the **standard representation** we write $u = (\phi, \chi)$, and with the explicit form of the $\gamma-$matrices in Eq. (3.19) the Dirac equation for $u(\boldsymbol{p})$ becomes

$$\begin{pmatrix} E_p - m & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -(E_p+m)\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0 \quad\Rightarrow\quad \begin{array}{rl}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\,\chi &= (E_p-m)\,\phi\,,\\ (\boldsymbol{p}\cdot\boldsymbol{\sigma})\,\phi &= (E_p+m)\,\chi\,.\end{array} \tag{3.70}$$

This is again consistent because $(\boldsymbol{p}\cdot\boldsymbol{\sigma})(\boldsymbol{p}\cdot\boldsymbol{\sigma}) = \boldsymbol{p}^2 = E_p^2 - m^2 = (E_p+m)(E_p-m)$, and the solution can be written as

$$u_s(\boldsymbol{p}) = \sqrt{E_p+m}\begin{pmatrix}\xi_s\\ \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E_p+m}\,\xi_s\end{pmatrix}, \qquad v_s(\boldsymbol{p}) = \sqrt{E_p+m}\begin{pmatrix}\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E_p+m}\,\eta_s\\ \eta_s\end{pmatrix}. \tag{3.71}$$

The standard representation is convenient because in the rest frame ($\boldsymbol{p} = 0$, $E_p = m$) only the upper component of $u_s(\boldsymbol{p})$ and the lower component of $v_s(\boldsymbol{p})$ survives, which correspond to the positive- and negative-energy eigenstates. Therefore it is also useful for describing a nonrelativistic particle with $v \ll c$ where the lower component of $u_s(\boldsymbol{p})$ can be neglected. This is the essential difference between the chiral representation, where the upper and lower components separate left- and right-handedness, and the standard representation where they are related to positive and negative energies.

Classical field theory vs. quantum mechanics. In the spirit of the scalar field we could equip the solutions of the Dirac equation with a scalar product,

$$\langle\psi_1,\psi_2\rangle := \int d\sigma_\mu\,\overline{\psi}_1(x)\,\gamma^\mu\psi_2(x) = \int d^3x\,\psi_1^\dagger(x)\,\psi_2(x)\,, \tag{3.72}$$

whose norm $\langle\psi|\psi\rangle$ is again the $U(1)$ charge. By doing so we entered relativistic quantum mechanics: if we interpret the field $\psi(x)$ as the wave function of a single particle, whose scalar product is Eq. (3.72), then the quantities H, $\boldsymbol{P}$ and $\widetilde{\boldsymbol{J}}$ in Eqs. (3.37) and (5.21) can be interpreted as the expectation values of the Hamilton operator $\boldsymbol{\gamma}\cdot(-i\boldsymbol{\nabla}) + m$, the momentum operator $-i\boldsymbol{\nabla}$, and the angular momentum operator $\boldsymbol{x}\times(-i\boldsymbol{\nabla}) + \frac{\boldsymbol{\Sigma}}{2}$, respectively. In that sense, quantum mechanics *is* essentially classical field theory, except that the additional scalar product also allows for a probability interpretation of the field. In contrast to the scalar field, the scalar product is indeed positive definite because when we insert the Dirac solutions the $U(1)$ charge takes the form

$$\langle \psi, \psi \rangle = \int d^3x \, \psi^\dagger(x) \, \psi(x) = \int \frac{d^3p}{2E_p} \sum_s \left(|a_{p,s}|^2 + |b_{p,s}|^2 \right) . \tag{3.73}$$

In exchange, the Hamiltonian is no longer positive definite and permits negative-energy eigenvalues.

In quantum field theory we omit the single-particle interpretation but rather view $\psi(x)$, $\overline{\psi}(x)$ as field operators on the Fock space. The quantities H, $\boldsymbol{P}$ and $\widetilde{\boldsymbol{J}}$ then become the Hamilton, momentum and angular-momentum operators of the field, and their eigenvalues are the total energy, momentum and angular momentum of some multiparticle state. After quantization with anticommutators, the situation above is also reversed: the Hamiltonian becomes positive but the $U(1)$ charge is no longer positive definite. This is no reason to worry because the charge is no longer interpreted as a probability; it is the number operator that counts the number of particles *minus* antiparticles in a state.

There is another piece of insight that we can take away from the discussion: since the structure of quantum mechanics is basically that of classical field theory, it reflects the 'classical' *tree-level* contributions to quantum processes, whereas loop corrections are reserved for the quantum field-theoretical treatment. In QED the electromagnetic coupling is so small that tree-level diagrams already provide a good approximation — which explains the successes of quantum mechanics in describing electrons, photons, and the physics of atoms and molecules.