8 Loops and renormalization

From the Feynman rules in the last section we know how to write down the diagrams that contribute to a given $n-$point function at some order in perturbation theory. Take for example the four-point function in ϕ^4 theory:

$$i\mathcal{M} = -i\lambda + \frac{(-i\lambda)^2}{2}\Big[\underbrace{\int\frac{d^4k}{(2\pi)^4}\,D_F(k)\,D_F(p-k)}_{=:\,\mathcal{A}(p)}+\text{perm.}\Big] + \mathcal{O}(\lambda^3)\,. \tag{8.1}$$

Here, $-i\lambda$ is the tree-level vertex and $\mathcal{A}(p)$ with $p = p_1 + p_2$ the amputated 1-loop diagram in Eq. (7.49) that leads to Eq. (7.53):

$$\mathcal{A}(p) = \int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m_0^2+i\epsilon}\,\frac{i}{(k-p)^2-m_0^2+i\epsilon}\,. \tag{8.2}$$

It depends on an external momentum p and we integrate over the loop momentum k. For $k^2 \to \infty$, the integral is proportional to d^4k/k^4, and therefore the integral diverges logarithmically.

The question is: how can we actually calculate such integrals and isolate the divergences that they contain? And after doing so, what should we do with them? It will turn out that the structure of 1-loop integrals is the same independently of the theory we are interested in, so eventually we can take over the results directly to QED.

Feynman parameters. The first step is a convenient trick based on the formula

$$\int_0^1 dx\,\frac{1}{[xa+(1-x)\,b\,]^2} = -\frac{1}{a-b}\,\frac{1}{xa+(1-x)\,b}\bigg|_0^1 = -\frac{1}{a-b}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{ab}\,, \tag{8.3}$$

which we can also write in the form

$$\frac{1}{ab} = \int_0^1 dx\int_0^1 dy\;\delta(x+y-1)\,\frac{1}{(xa+yb)^2}\,, \tag{8.4}$$

where $x, y \in [0,1]$ are called Feynman parameters. More generally,

$$\frac{1}{a_1\dots a_n} = \int dx_1\dots dx_n\,\delta\left(\sum_{i=1}^n x_i - 1\right)\frac{(n-1)!}{\left[\sum\limits_{i=1}^n x_i\,a_i\right]^n}\,. \tag{8.5}$$

The structure of loop integrals is always that of Eq. (8.2), with one loop momentum k and one or several external momenta p_i, and possibly more than just two internal propagators. Let's evaluate the formula specifically for $a_i = (k+p_i)^2 - m_i^2 + i\epsilon$. In that case

$$\sum_i x_i\,a_i = \sum_i x_i\left(k^2+p_i^2+2k\cdot p_i - m_i^2 + i\epsilon\right) = k^2 + \sum_i x_i\left(2k\cdot p_i + p_i^2 - m_i^2\right) + i\varepsilon\,,$$

where we exploited the constraint $\sum_i x_i = 1$ that is imposed by the δ−function. Now, define a new loop momentum l via

$$l = k + \sum_i x_i\, p_i \qquad \Rightarrow \qquad l^2 = k^2 + 2\sum_i x_i\, k\cdot p_i + \Big(\sum_i x_i\, p_i\Big)^2, \tag{8.6}$$

which leads to

$$\sum_i x_i\, a_i \;=\; l^2 - \Bigg[\underbrace{\Big(\sum_i x_i\, p_i\Big)^2 - \sum_i x_i\,(p_i^2 - m_i^2)}_{=:\,\Delta}\Bigg] + i\epsilon \;=\; l^2 - \Delta + i\epsilon\,. \tag{8.7}$$

The quantity Δ no longer depends on the loop momentum l. The expression (8.2) corresponds to $n = 2$; the resulting integrand only depends on l^2:

$$\mathcal{A}(p) = -\int_0^1 dx \int \frac{d^4 l}{(2\pi)^4}\,\frac{1}{(l^2 - \Delta + i\epsilon)^2}\,. \tag{8.8}$$

In that case $p_1 = -p$, $p_2 = 0$ and $m_1 = m_2 = m_0$, and therefore $l = k - xp$ and

$$\Delta \;=\; x^2 p^2 - xp^2 + xm_0^2 + (1-x)m_0^2 \;=\; m_0^2 - x(1-x)\,p^2\,. \tag{8.9}$$

Wick rotation. The pole structure of $\mathcal{A}(p)$ is the same as that for a single propagator: when we split the integral $\int d^4 l = \int d^3 l \int dl_0$, the bracket in the denominator gives

$$l^2 - \Delta + i\epsilon = l_0^2 - (\boldsymbol{l}^2 + \Delta) + i\epsilon\,, \tag{8.10}$$

with the same Feynman prescription for the integration contour: integrate below the pole at negative l_0 and above the pole at positive l_0. Since there are no further poles in the complex l_0 plane, we can equally deform the integration contour to follow the imaginary axis (Wick rotation) and define a **Euclidean momentum** l_E^μ:

$$l^0 = i l_E^0\,, \quad \boldsymbol{l} = \boldsymbol{l}_E \qquad \Rightarrow \qquad l^2 = -(l_E^0)^2 - \boldsymbol{l}_E^2 = -l_E^2\,, \quad d^4 l = i d^4 l_E\,. \tag{8.11}$$

The integral then becomes

$$\mathcal{A}(p) = -i\int_0^1 dx\, I_2^{(4)}\,, \qquad I_2^{(4)} := \int \frac{d^4 l_E}{(2\pi)^4}\,\frac{1}{(l_E^2 + \Delta)^2}\,, \tag{8.12}$$

where the subscript '2' is the power of the denominator and the superscript '(4)' the number of spacetime dimensions. For general loop integrals we arrive at the formula

$$\int \frac{d^4 k}{(2\pi)^4}\,\frac{1}{\prod_i \big[(k+p_i)^2 - m_i^2 + i\epsilon\big]} = i(-1)^n (n-1)! \int dx_1 \dots dx_n\, \delta\Big(\sum_i x_i - 1\Big)\, I_n^{(4)}\,, \tag{8.13}$$

with l defined in Eq. (8.6) and Δ in Eq. (8.7).

Regularization. Next, we want to calculate the integral $I_2^{(4)}$ explicitly. To do so, we write the four-dimensional integral as

$$d^4 l_E = dl_E \, l_E^3 \, d\Omega_4 = \frac{1}{2} \, dl_E^2 \, l_E^2 \, d\Omega_4 \,, \tag{8.14}$$

where $d\Omega_4$ is the four-dimensional unit sphere and $\int d\Omega_4 = 2\pi^2$. Hence we are left with a radial integral

$$I_2^{(4)} = \frac{1}{(4\pi)^2} \int_0^\infty dl_E^2 \, \frac{l_E^2}{(l_E^2 + \Delta)^2} \,, \tag{8.15}$$

which diverges logarithmically when $l_E^2 \to \infty$.

The idea is to make the integral finite by introducing some regulator, which will also isolate the components that diverge once the regulator is removed. There are several possible ways of regularization. We will discuss three of them here: a momentum cutoff, Pauli-Villars regularization, and dimensional regularization. There are also other well-established methods such as lattice regularization, proper-time regularization etc.

UV momentum cutoff. Since the divergence is produced by the UV momentum modes, the simplest strategy is to impose a hard cutoff: we do not integrate l_E^2 over the full momentum range but only up to a cutoff $l_E^2 < \Lambda^2$. Setting $l_E^2 = z$, the integral becomes:

$$\begin{aligned} \int_0^{\Lambda^2} dz \, \frac{z}{(z+\Delta)^2} &= \int_0^{\Lambda^2} dz \left[\frac{z+\Delta}{(z+\Delta)^2} - \frac{\Delta}{(z+\Delta)^2} \right] = \left[\ln(z+\Delta) + \frac{z}{z+\Delta} \right]_0^{\Lambda^2} \\ &= \ln\left(\frac{\Lambda^2+\Delta}{\Delta}\right) + \frac{\Delta}{\Lambda^2+\Delta} - 1 \quad \xrightarrow{\Lambda\to\infty} \quad \ln\frac{\Lambda^2}{\Delta} \,. \end{aligned} \tag{8.16}$$

In the context of QED we will later see that a cutoff regularization breaks gauge invariance, so it is not the most suitable method to use. In practice it is more convenient to use dimensional regularization or Pauli-Villars regularization which both preserve gauge invariance.[5]

Pauli-Villars regularization. The idea of Pauli-Villars regularization is to modify one of the propagators in the loop integral so that the integrand vanishes faster in the ultraviolet. To do so, we start from the original expression (8.2), where we subtract another propagator with a large mass $\sqrt{m_0^2 + \Lambda^2}$:

$$\frac{1}{k^2 - m_0^2} \quad \to \quad \frac{1}{k^2 - m_0^2} - \frac{1}{k^2 - m_0^2 - \Lambda^2} = \frac{1}{k^2 - m_0^2} \, \frac{1}{1 - \frac{k^2 - m_0^2}{\Lambda^2}} \,. \tag{8.17}$$

Therefore, the propagator now vanishes as $\sim 1/k^4$ for $k^2 \to \infty$, and the integrand with a power $\sim 1/k^6$. The remaining steps up to Eq. (8.15) go through as before, but we

[5] One should keep in mind, however, that in the course of a numerical evaluation of loop integrals, where the momentum integration becomes a discretized sum, one always introduces a hard cutoff because a computer cannot integrate up to infinity. In that case one has to be especially careful about potential gauge artifacts.

have to subtract $I_2^{(4)} - I_2'^{(4)}$, where $I_2'^{(4)}$ is obtained from setting

$$\begin{aligned} p_1 &= -p\,, & m_1 &= \sqrt{m_0^2+\Lambda^2}\,, \\ p_2 &= 0\,, & m_2 &= m_0 \end{aligned} \qquad \Rightarrow \qquad \Delta' = \Delta + x\Lambda^2 \tag{8.18}$$

in the Feynman parameter representation. Then we get

$$\int_0^\infty dl_E^2 \left[\frac{l_E^2}{(l_E^2+\Delta)^2} - \frac{l_E^2}{(l_E^2+\Delta+x\Lambda^2)^2} \right] \mathrel{\dot{\dot{=}}} \ln\left(1+\frac{x\Lambda^2}{\Delta}\right) \tag{8.19}$$

which for $\Lambda \to \infty$ diverges again logarithmically.

Dimensional regularization. The most common regularization method in the context of perturbation theory is dimensional regularization. Here the idea is to first calculate the integral in d dimensions and take the limit $d \to 4$ in the end. We write

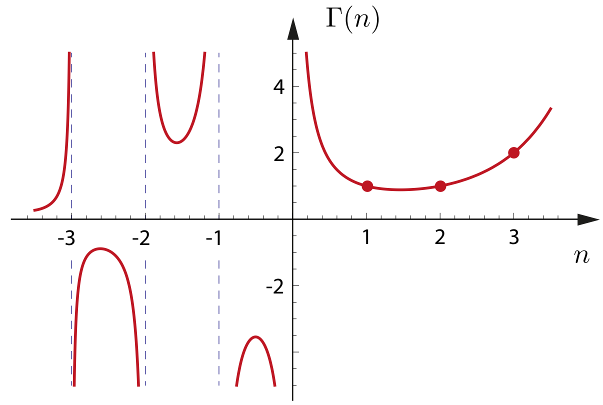
$$I_2^{(d)} = \frac{1}{M^{d-4}} \int \frac{d^d l_E}{(2\pi)^d} \, \frac{1}{(l_E^2+\Delta^2)^2}\,, \tag{8.20}$$

where the factor M is an arbitrary mass scale that we introduced to ensure that the integral remains dimensionless also in d spacetime dimensions. Its origin is the dimension of the coupling constant in front of the integral: for a ϕ^4 theory in four dimensions, λ is dimensionless but this is no longer the case for arbitrary d. The volume integral becomes

$$d^d l_E = dl_E\, l_E^{d-1}\, d\Omega_d = \frac{1}{2}\, dl_E^2\, (l_E^2)^{\frac{d}{2}-1}\, d\Omega_d\,, \qquad \int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}\big)}\,, \tag{8.21}$$

where $d\Omega_d$ is the unit sphere in d dimensions. $\Gamma(n)$ is the Gamma function; let us recall a few of its properties:

- $\Gamma(n) = \int_0^\infty dx\, x^{n-1} e^{-x}$,
- $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}_+$,
- $\Gamma(n)$ has poles at $n = 0, -1, -2, \dots$
- $\Gamma(n+1) = n\,\Gamma(n)$,
- $\Gamma'(1) = -\gamma = -0.5772\dots$ is the Euler-Mascheroni constant.



It is easy to prove the result (8.21) for $\int d\Omega_d$:

$$\begin{aligned} (\sqrt{\pi})^d &= \left(\int_{-\infty}^{\infty} dx\, e^{-x^2} \right)^d = \int d^d x\, e^{-\sum\limits_{i=1}^{d} x_i^2} \\ &= \frac{1}{2} \int dx^2 (x^2)^{d/2-1}\, e^{-x^2} \int d\Omega_d = \frac{1}{2}\,\Gamma\!\left(\frac{d}{2}\right) \int d\Omega_d\,. \end{aligned} \tag{8.22}$$

Now, take the integral (8.20) and insert Eq. (8.21):

$$I_2^{(d)} = \frac{1}{M^{d-4}} \frac{\pi^{d/2}}{(2\pi)^d} \frac{1}{\Gamma\big(\frac{d}{2}\big)} \int_0^\infty dl_E^2 \, \frac{(l_E^2)^{d/2-1}}{(l_E^2+\Delta)^2} \,. \tag{8.23}$$

With the substitution

$$z = \frac{\Delta}{l_E^2+\Delta} \qquad \Rightarrow \qquad dz = -dl_E^2 \, \frac{\Delta}{(l_E^2+\Delta)^2} \,, \quad l_E^2 = \frac{\Delta}{z}\,(1-z) \tag{8.24}$$

we can transform it into

$$I_2^{(d)} = \frac{1}{M^{d-4}} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma\big(\frac{d}{2}\big)} \left(\frac{1}{\Delta}\right)^{2-d/2} \underbrace{\int_0^1 dz\, z^{1-d/2}\,(1-z)^{d/2-1}}_{=\mathcal{B}\left(2-\frac{d}{2},\frac{d}{2}\right) = \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(2)}} \,. \tag{8.25}$$

We expressed the remaining integral through Euler's Beta function

$$\mathcal{B}(m,n) = \int_0^1 dx\, x^{m-1}\,(1-x)^{n-1} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \,, \tag{8.26}$$

so that we arrive at the result

$$I_2^{(d)} = \frac{1}{M^{d-4}} \frac{\Gamma\big(2-\frac{d}{2}\big)}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-d/2} \,. \tag{8.27}$$

This expression diverges for $d = 4, 6, 8, \dots$ but is otherwise well-defined, even if d is non-integer. Hence, we can use it as a *definition* of the original integral for non-integer dimensions.

In the final step we set $d = 4 - \varepsilon$,

$$I_2^{(d)} = M^\varepsilon \, \frac{\Gamma\big(\frac{\varepsilon}{2}\big)}{(4\pi)^{2-\varepsilon/2}} \left(\frac{1}{\Delta}\right)^{\varepsilon/2} = \frac{\Gamma\big(\frac{\varepsilon}{2}\big)}{(4\pi)^2} \left(\frac{4\pi M^2}{\Delta}\right)^{\varepsilon/2} \,, \tag{8.28}$$

and expand the expression around $\varepsilon = 0$. Using $x^{\varepsilon/2} = e^{\frac{\varepsilon}{2}\ln x} = 1 + \frac{\varepsilon}{2}\ln x + \mathcal{O}(\varepsilon^2)$ and $\Gamma\big(\frac{\varepsilon}{2}\big) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)$, we find

$$I_2^{(d)} = \frac{1}{(4\pi)^2} \left[\frac{2}{\varepsilon} - \gamma + \ln\left(\frac{4\pi M^2}{\Delta}\right) + \mathcal{O}(\varepsilon) \right]. \tag{8.29}$$

The integral has a part $\sim 1/\varepsilon$ that diverges for $\varepsilon \to 0$, and a remainder that is finite and depends on M, which is completely arbitrary because it was only introduced for dimensional reasons. In principle we could also combine the finite parts since $-\gamma = \ln e^{-\gamma}$ and write

$$-\gamma + \ln\left(\frac{4\pi M^2}{\Delta}\right) = \ln\left(\frac{4\pi M^2\, e^{-\gamma}}{\Delta}\right) = \ln \frac{\widetilde{M}^2}{\Delta}. \tag{8.30}$$

The finite parts have formally the same structure as for cutoff and Pauli-Villars regularization, because also in those cases we can always introduce a mass scale $\widetilde{M}$ such that for $\Lambda \to \infty$

$$\ln \frac{x\Lambda^2}{\Delta} = \ln \frac{x\Lambda^2}{\widetilde{M}^2} + \ln \frac{\widetilde{M}^2}{\Delta} \,. \tag{8.31}$$

The divergent terms differ, however: they may diverge logarithmically with $\ln \Lambda^2$, or with $1/\varepsilon$ as in dimensional regularization.

In complete analogy one can also work out the following integrals:

$$\begin{aligned} I_n^{(d)} &= \int \frac{d^d l_E}{(2\pi)^d} \, \frac{1}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \, \frac{\Gamma\big(n-\frac{d}{2}\big)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \,, \\ \widetilde{I}_n^{(d)} &= \int \frac{d^d l_E}{(2\pi)^d} \, \frac{l_E^2}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \, \frac{d}{2} \, \frac{\Gamma\big(n-\frac{d}{2}-1\big)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \,. \end{aligned} \tag{8.32}$$

In summary, the expression for $\mathcal{A}(p)$ in Eq. (8.1), using Eqs. (8.12) and (8.29), becomes

$$\mathcal{A}(p) = -i \int_0^1 dx \, I_2^{(4)} = -\frac{i}{(4\pi)^2} \, \lim_{\varepsilon\to 0} \int_0^1 dx \left[\frac{2}{\varepsilon} - \gamma + \ln\left(\frac{4\pi M^2}{\Delta}\right)\right] . \tag{8.33}$$

A common feature of all regularization methods is that they always introduce a **scale** M in the theory, which remains there even if we formally remove the divergent terms. This new scale dependence has profound consequences: even if the mass parameter in the Lagrangian is zero and the classical theory is scale invariant, the renormalized quantum field theory is *not* because in the process of regularization we have picked up a scale. Classical symmetries that are broken at the quantum level are called **anomalous**, so this effect is also called the 'anomalous breaking of scale invariance'.

Renormalization. So we can calculate loop diagrams explicitly by introducing some regulator, and we can separate the finite parts from the divergent ones. The ultimate question is: what should we do with the divergences? Should we simply throw them away, and if yes, how would that make any sense? Surprisingly enough, this is indeed what eventually has to happen, but there is a deeper underlying reason which can be understood in the course of renormalization. The idea is the following: let's interpret all fields, masses and couplings that appear in the Lagrangian as 'bare' and unphysical, and write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\Phi_\mathrm{B}\,\partial^\mu\Phi_\mathrm{B} - \frac{1}{2}\,m_\mathrm{B}^2\,\Phi_\mathrm{B}^2 - \frac{\lambda_\mathrm{B}}{4!}\,\Phi_\mathrm{B}^4 \stackrel{\text{p.I.}}{=} -\frac{1}{2}\,\Phi_\mathrm{B}\,(\Box + m_\mathrm{B}^2)\,\Phi_\mathrm{B} - \frac{\lambda_\mathrm{B}}{4!}\,\Phi_\mathrm{B}^4 \tag{8.34}$$

with a subscript 'B' for bare. Now define a **renormalized** field Φ, renormalized mass m and renormalized coupling λ by

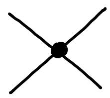
$$\Phi_\mathrm{B} = Z_\phi^{1/2}\,\Phi\,, \qquad m_\mathrm{B}^2 = Z_m\,m^2\,, \qquad \lambda_\mathrm{B} = Z_\lambda\,\lambda\,, \tag{8.35}$$

where we introduced three **renormalization constants** Z_ϕ, Z_m and Z_λ. They are, as of now, undetermined and potentially divergent. Consequently, the Lagrangian takes the form[6]

$$\mathcal{L} = -\frac{1}{2} Z_\phi \, \Phi \, (\Box + Z_m \, m^2) \, \Phi - Z_\lambda \, Z_\phi^2 \, \frac{\lambda}{4!} \, \Phi^4 \,. \tag{8.36}$$

Since we can read off the tree-level propagators and vertices from the Lagrangian, the renormalization constants will also enter in their Feynman rules. We will call them the *renormalized tree-level* propagator and vertex:

$$D_0(p) = \frac{i}{Z_\phi \, (p^2 - Z_m \, m^2)} \quad \Leftrightarrow \quad i D_0^{-1}(p) = Z_\phi \, (p^2 - Z_m \, m^2) \,,$$



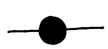
$$\Gamma_0(\{p_i\}) = -i\lambda \, Z_\lambda \, Z_\phi^2 \,. \tag{8.37}$$

The $i\epsilon$ prescription is still intact but we drop it for brevity.

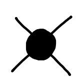
Consider now the **full 1PI Green functions** of the theory. The set of all 1PI functions defines the quantum field theory completely because the effective action can be expressed by them (we might return to this at some later point). We have seen that we can reconstruct the propagator from its 1PI counterpart, Eq. (7.42), and generally this is true for all S-matrix elements: the connected, amputated S-matrix elements can be expressed in terms of 1PI Green functions together with dressed propagator insertions. The 1PI n-point functions are also convenient for the discussion of renormalization as we will see shortly. If we denote the full propagator by $D(p)$, then it is related to the 1PI self-energy via

$$D(p) = D_0 + D_0 \, \frac{\Sigma}{i} \, D_0 + \dots = D_0 \, (1 + i\Sigma \, D) \quad \Rightarrow \quad i D^{-1} = i D_0^{-1} - \Sigma \,, \tag{8.38}$$

and so we can generally write

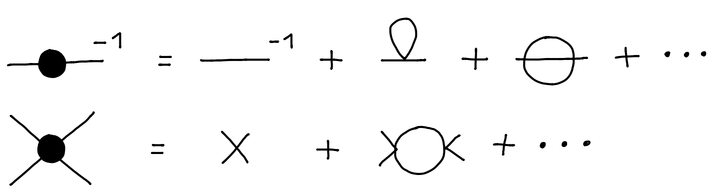


$$i D^{-1}(p) = Z_\phi \, (p^2 - Z_m \, m^2) - \Sigma(p) \,,$$



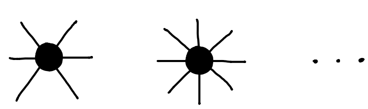
$$\Gamma(\{p_i\}) = -i\lambda \, Z_\lambda \, Z_\phi^2 + i\Omega(\{p_i\}) \,. \tag{8.39}$$

$\Sigma(p)$ defines the self-energy as before, and its analogue for the four-point function is Ω: it contains all 1PI loop diagrams that we can draw order by order in perturbation theory. In terms of Feynman diagrams:



(8.40)

In principle the list goes on for the six-point function, eight-point function, etc.,



(8.41)

[6]This way of discussing renormalization is also called 'renormalized perturbation theory'. The alternative is 'bare perturbation theory' which is completely equivalent but somewhat more confusing, so we will not discuss it here.

except that they do not have tree-level contributions but start off with loop diagrams right away. In ϕ^4 theory there are also no $n-$point functions with an odd number of legs; this is due to the invariance of the Lagrangian under $\phi \to -\phi$.

The idea is now that the full propagator should have a pole at $p^2 = m^2$, where it corresponds to a free particle with mass m. Likewise, the full vertex should become a free vertex if its external legs are onshell:

$$D(p) \xrightarrow{p^2=m^2} \frac{i}{p^2-m^2}\,, \qquad \Gamma(\{p_i\}) \xrightarrow{p_i^2=m^2} -i\lambda\,. \tag{8.42}$$

Here, m and λ are the physical, measurable mass and coupling constant of the theory. Actually these **renormalization conditions** are completely arbitrary, so it makes sense to generalize them to some arbitrary **renormalization poin**t $p^2 = \mu^2$. This is especially practical in theories where the propagator does *not* have a Källén-Lehmann representation. An example is QCD, where there are no free quarks due to confinement. Hence we demand

$$iD^{-1}(p)\Big|_{p^2=\mu^2} \stackrel{!}{=} p^2 - m^2\,, \quad \frac{d}{dp^2}\,iD^{-1}(p)\Big|_{p^2=\mu^2} \stackrel{!}{=} 1\,, \quad \Gamma(\{p_i\})\Big|_{p_i^2=\mu^2} \stackrel{!}{=} -i\lambda\,. \tag{8.43}$$

The first condition fixes the 'pole position' through the mass m (which is a true pole only if $\mu = m$), the second sets the residue at the pole, and the third fixes the coupling constant. Now let's insert this into Eq. (8.39). If we abbreviate

$$\Sigma(p)\Big|_{p^2=\mu^2} = \Sigma_\mu\,, \qquad \frac{d}{dp^2}\,\Sigma(p)\Big|_{p^2=\mu^2} = \Sigma'_\mu\,, \qquad \Omega(\{p_i\})\Big|_{p_i^2=\mu^2} = \Omega_\mu \tag{8.44}$$

we arrive at

$$\begin{aligned} Z_\phi\,(\mu^2 - Z_m m^2) - \Sigma_\mu = \mu^2 - m^2 \qquad &\Rightarrow \qquad Z_\phi\, Z_m = 1 + \frac{\mu^2\,\Sigma'_\mu - \Sigma_\mu}{m^2}\,, \\ Z_\phi - \Sigma'_\mu = 1 \qquad &\Rightarrow \qquad Z_\phi = 1 + \Sigma'_\mu\,, \\ -i\lambda\, Z_\lambda\, Z_\phi^2 + i\Omega_\mu = -i\lambda \qquad &\Rightarrow \qquad Z_\phi^2\, Z_\lambda = 1 + \frac{\Omega_\mu}{\lambda}\,. \end{aligned} \tag{8.45}$$

These conditions determine the three renormalization constants: at lowest order perturbation theory they are all equal to one, whereas at higher orders they pick up loop contributions from Σ_μ, Σ'_μ and Ω_μ which have divergent and finite parts. Hence their generic structure is of the form

$$Z_i(\lambda, m, \epsilon) = 1 + \sum_{k=1}^{\infty} c_k(\lambda, m, \epsilon)\,\lambda^k\,, \tag{8.46}$$

with divergent coefficients c_k. On the other hand, when we substitute this back into Eq. (8.39) we find

$$\begin{aligned} iD^{-1}(p) &= (1+\Sigma'_\mu)\,p^2 - m^2 - \mu^2\,\Sigma'_\mu + \Sigma_\mu - \Sigma(p) \\ &= p^2 - m^2 - (\Sigma(p) - \Sigma_\mu) + (p^2 - \mu^2)\,\Sigma'_\mu\,, \\ \Gamma(\{p_i\}) &= -i\lambda + i\,(\Omega(\{p_i\}) - \Omega_\mu)\,. \end{aligned} \tag{8.47}$$

The crucial point is that by means of the subtraction at the renormalization point the divergences *cancel* in the renormalized Green functions. Therefore, the renormalized n-point functions are finite!

Let's have a look at a concrete example, namely the one-loop contribution to the four-point function. We have worked out its structure earlier; the result in dimensional regularization was Eq. (8.33):

$$\Omega(\{p_i\}) = \frac{\lambda^2}{2} \int_0^1 dx\, I_2^{(4)} = \frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \int_0^1 dx \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{\Delta} \right], \tag{8.48}$$

with $\Delta = m_B^2 - x(1-x)\,p^2$ and $p = p_1 + p_2$, plus the two permutations which we do not write explicitly. In principle, by means of the Feynman rules (8.37) the diagram picks up an additional prefactor

$$\frac{Z_\lambda^2\, Z_\phi^4}{Z_\phi^2} = Z_\lambda^2\, Z_\phi^2 = 1 + \mathcal{O}(\lambda)\,, \tag{8.49}$$

but since the correction comes with powers of the coupling constant it will only contribute at higher orders in perturbation theory, so we can ignore it in the one-loop result. For simplicity we renormalize the four-point function at $p^2 = (p_1+p_2)^2 = \mu^2$. Observe that the subtraction cancels the divergent piece $\sim 1/\epsilon$:

$$\begin{aligned} \Omega(\{p_i\}) - \Omega_\mu &= \frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \int_0^1 dx\, \ln \frac{\Delta_\mu}{\Delta} = \frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \int_0^1 dx\, \ln \frac{m_\mathrm{B}^2 - x(1-x)\,\mu^2}{m_\mathrm{B}^2 - x(1-x)\,p^2} \\ &= \frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \int_0^1 dx\, \ln \frac{m^2 - x(1-x)\,\mu^2}{m^2 - x(1-x)\,p^2}\,. \end{aligned} \tag{8.50}$$

In the last step we have used that $m_\mathrm{B}^2 = Z_m\, m^2 = m^2 + O(\lambda)$, so the correction will also only appear at higher orders and to lowest order we can set $m_\mathrm{B} = m$. The resulting expression depends on the renormalized mass m and coupling λ. It is finite, but in turn it depends now on the arbitrary renormalization point μ.

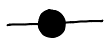
Counterterms. It is customary to write the renormalization constants as

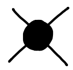
$$Z_\phi = 1 + \delta Z_\phi\,, \qquad Z_m\, Z_\phi = 1 + \frac{\delta m^2}{m^2}\,, \qquad Z_\lambda\, Z_\phi^2 = 1 + \frac{\delta_\lambda}{\lambda}\,. \tag{8.51}$$

In that way the Lagrangian (8.34) can be split into a piece that depends only on renormalized quantities, plus a counterterm that includes the new 'renormalization constants' δZ_ϕ, δm^2 and $\delta\lambda$:

$$\mathcal{L} = -\frac{1}{2}\,\Phi\,(\Box + m^2)\,\Phi - \frac{\lambda}{4!}\,\Phi^4 - \frac{1}{2}\,\Phi\,(\delta Z_\varphi\,\Box + \delta m^2)\,\Phi - \frac{\delta\lambda}{4!}\,\Phi^4\,. \tag{8.52}$$

The counterterms can be interpreted as new tree-level propagators and vertices with corresponding Feynman rules. This is especially convenient for calculating higher loops, because eventually it would become hard to keep track of the Z_i factors in front of the integrals from lower orders in perturbation theory (which we can ignore for one-loop graphs). Instead, one must now systematically add diagrams with 'counter' propagators and vertices. The expressions (8.39) for the full 1PI Green functions become





$$\begin{aligned} iD^{-1}(p) &= p^2 - m^2 - \Sigma(p) + \delta Z_\phi \, p^2 - \delta m^2 \,, \\ i\Gamma(\{p_i\}) &= \lambda - \Omega(\{p_i\}) + \delta\lambda \,, \end{aligned} \tag{8.53}$$

i.e., the new renormalization constants can be directly identified with the counterterms that cancel the singularities. If we apply our earlier renormalization conditions and compare Eq. (8.51) with (8.45) we find

$$\delta Z_\phi = \Sigma'_\mu \,, \qquad \delta m^2 = \mu^2 \, \Sigma'_\mu - \Sigma_\mu \,, \qquad \delta\lambda = \Omega_\mu \,. \tag{8.54}$$

Renormalization schemes. The examples discussed so far highlight some general features of renormalization:

- If a given theory contains a finite number of renormalization constants Z_i (three in ϕ^4 theory), we must specify equally many renormalization conditions to determine them. This in turn removes all UV divergences from the theory. We will provide more detailed arguments below.
- All physical quantities are independent of ϵ and Z_i and they are finite. The Lagrangian $\mathcal{L}$ itself is divergent, but this is irrelevant because it is not an observable.
- The mass $m(\mu)$ and coupling $\lambda(\mu)$ depend now on the renormalization point μ, where they are specified as an external input. That is, they are parameters of the theory and can no longer be determined *within* the theory — they must be taken from experiment.

In QED we can use **onshell renormalization** with $\mu^2 = m^2$. The electron propagator has a pole at $p^2 = m^2$, where m is the physical mass of the electron. The photon is massless, so its propagator has a pole at $q^2 = 0$. This is where one can match the coupling constant (the electron charge) with experiment, because two infinitely separated charges correspond to a propagator evaluated at $q^2 = 0$. On the other hand, onshell renormalization doesn't work in QCD because there are no free quarks and gluons due to confinement. As a consequence, the quark masses and the coupling have to be specified at some suitable renormalization scale where theory predictions can be compared to experiment.[7]

The arbitrariness in the specification of $m(\mu)$ and $\lambda(\mu)$ is reflected in the **renormalization scheme**. Imposing *overall* renormalization conditions of the form (8.43) on the Green functions defines a momentum subtraction (MOM) scheme. This is convenient for nonperturbative calculations since at no point in the previous discussion

[7] This scale should also be spacelike ($\mu^2 < 0$ in Minkowski conventions) to avoid branch-cut singularities that appear in the loop diagrams. High-energy scattering experiments with hadrons probe the domain of large spacelike momenta of internal quarks and gluons, which is also where the QCD coupling is small and perturbation theory applicable.

we *needed* to resort to a perturbative expansion: Eqs. (8.39) can be equally viewed as Dyson-Schwinger equations (cf. Eq. (7.46)) which are nonperturbative and exact. Alternatively, one can also explicitly subtract only the divergent terms order by order in perturbation theory, such as the one $\sim 1/\epsilon$ in Eq. (8.48), which defines the MS scheme (minimal subtraction). In that case our definition of the renormalization scale μ is no longer available; instead, the scale $M \equiv \mu$ takes its place as it doesn't get cancelled by the subtraction anymore. (In the MOM scheme, we have essentially traded the dependence on M by a dependence on μ.) Another possibility is to subtract not only the divergences but *all* terms that are not explicitly dependent on $M \equiv \mu$; this defines the $\overline{\text{MS}}$ scheme (modified minimal subtraction).

As a consequence, the masses and couplings depend not only on the renormalization point but also on the renormalization scheme, and the different schemes are related to each other by finite constants:

$$\begin{matrix} m(\mu)_{\text{MOM}} \\ \lambda(\mu)_{\text{MOM}} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} m(\mu)_{\text{MS}} \\ \lambda(\mu)_{\text{MS}} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} m(\mu)_{\overline{\text{MS}}} \\ \lambda(\mu)_{\overline{\text{MS}}} \end{matrix} \quad \leftrightarrow \quad \dots \tag{8.55}$$

The Green functions themselves depend on the renormalization point μ, but they are independent of the scheme. For example:

$$D\left(p, \mu, m(\mu)_{\text{MOM}}, \lambda(\mu)_{\text{MOM}}\right) = D\left(p, \mu, m(\mu)_{\overline{\text{MS}}}, \lambda(\mu)_{\overline{\text{MS}}}\right) = \dots \tag{8.56}$$

The invariance of measurable quantities under a change of μ and different renormalization schemes leads to the concept of the **renormalization group**.

As an example, consider the 1PI four-point function and write it with counterterms as in Eq. (8.53):

$$\begin{aligned} i\Gamma(p) &= \lambda - \Omega(p) + \delta\lambda = \lambda - \frac{\lambda^2}{32\pi^2}\int dx \left[\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{\Delta}\right] + \delta\lambda \\ &= \lambda - \frac{\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{m^2} - \int dx\, \ln\left(1 - x(1-x)\,\frac{p^2}{m^2}\right)\right] + \delta\lambda\,. \end{aligned} \tag{8.57}$$

For simplicity we ignore again the contribution from the two permuted diagrams, so the expression depends only on the $s-$channel momentum $p = p_1 + p_2$. In the MOM scheme we impose the condition

$$i\Gamma(p)_{p^2=\mu^2} \stackrel{!}{=} \lambda_{\text{MOM}}\,, \tag{8.58}$$

which leads to

$$\delta\lambda_{\text{MOM}} = \frac{\lambda_{\text{MOM}}^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{m^2} - \int dx\, \ln\left(1 - x(1-x)\,\frac{\mu^2}{m^2}\right)\right]. \tag{8.59}$$

In the MS and $\overline{\text{MS}}$ scheme we do not impose such a condition but instead subtract terms by hand. In MS we would only subtract the divergent term, whereas in $\overline{\text{MS}}$ we also subtract the remaining $\mu-$independent terms:

$$\delta\lambda_{\text{MS}} = \frac{\lambda_{\text{MS}}^2}{32\pi^2}\,\frac{2}{\epsilon}\,, \qquad \delta\lambda_{\overline{\text{MS}}} = \frac{\lambda_{\overline{\text{MS}}}^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln 4\pi\right]. \tag{8.60}$$

In any case, whatever we decide to do cannot change the four-point function, which must remain the same. For example evaluated at the renormalization point:

$$i\Gamma(p)_{p^2=\mu^2} = \lambda_{\text{MOM}} = \lambda_{\overline{\text{MS}}} - \frac{\lambda_{\overline{\text{MS}}}^2}{32\pi^2}\left[\ln\frac{\mu^2}{m_{\overline{\text{MS}}}^2} - \int dx\, \ln\left(1 - x(1-x)\,\frac{\mu^2}{m_{\overline{\text{MS}}}^2}\right)\right], \tag{8.61}$$

which gives us the relation between λ_{MOM} and $\lambda_{\overline{\text{MS}}}$.

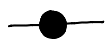
Renormalizability. So far we have only considered one explicit diagram. Do the singularities *always* cancel? Let's consider the action for a generic ϕ^p theory:

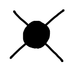
$$S = -\int d^4x \left[\frac{1}{2}\,\Phi\,(\Box + m^2)\Phi + \frac{\lambda}{p!}\,\Phi^p \right], \tag{8.62}$$

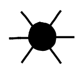
where we suppress the renormalization constants for simplicity. Now count the mass dimensions of the quantities that appear in the action:

$$[S] = 0 \qquad \Rightarrow \qquad [\mathcal{L}] = 4\,, \quad [\Phi] = 1\,, \quad [\Phi^p] = p\,, \quad [\lambda] = 4 - p\,. \tag{8.63}$$

From here we can infer the dimensions of the 1PI Green functions in momentum space:

$\Gamma_2 =$  $= \; p^2 - m^2 + \ldots \quad \Rightarrow \quad [\Gamma_2] = 2\,,$

$\Gamma_4 =$  $\overset{\phi^4}{=} \; -i\lambda + \ldots \quad \Rightarrow \quad [\Gamma_4] = 0\,,$ (8.64)

$\Gamma_6 =$  $\overset{\phi^6}{=} \; -i\lambda + \ldots \quad \Rightarrow \quad [\Gamma_6] = -2\,.$

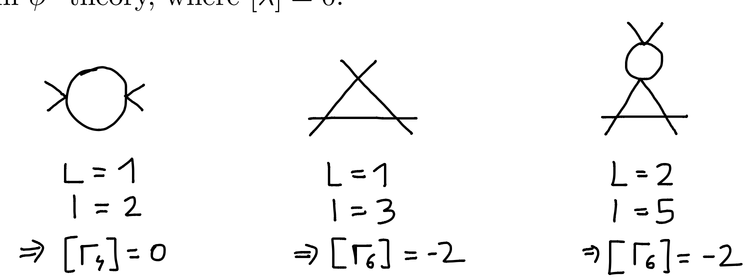
Remember from Eq. (7.33) that the tree-level vertex is always of the form $-i\lambda$ as long as λ is the corresponding ϕ^4, ϕ^6, ... coupling constant. That is, in a ϕ^4 theory the six-point function does not have a tree-level term, in a ϕ^6 theory the four-point function does not have a tree-level term, etc. In any case, the dimension of Γ_n is always the same independently of p, because it is already determined by $-i\lambda$:

$$[\Gamma_n] = 4 - n\,. \tag{8.65}$$

On the other hand, we can also count the dimension of a given n-point function by going into some order in perturbation theory. In that case, we would count the number of loops L (each comes with dimension four), the number of internal propagators I (each comes with dimension -2), and the number of vertices (where each has dimension $[\lambda]$). Therefore:

$$[\Gamma_n] = 4L - 2I + [\lambda]\,V\,. \tag{8.66}$$

For example in ϕ^4 theory, where $[\lambda] = 0$:



(8.67)

Obviously this is consistent.

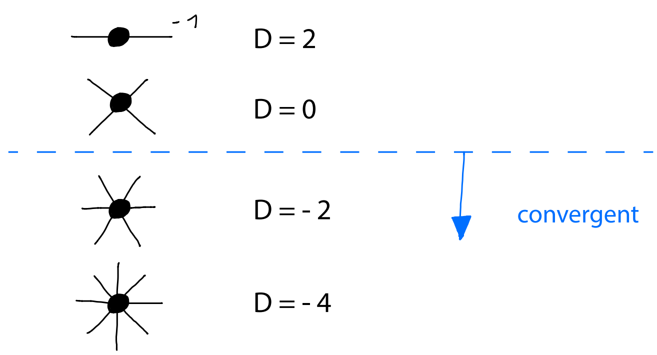
Now, the quantity $D = 4L - 2I$ also tells us how badly divergent a given diagram will be: if the number of loops L beats the number of propagators I it will diverge; if there are many propagators in a loop it will converge. D is called the **superficial degree of divergence**: if $D < 0$ the diagram converges, if $D \geq 0$ it will diverge. The first diagram above has $D = 0$ and diverges logarithmically. The second has $D = -2$ and is convergent; the third has $D = -2$ but unfortunately it is still divergent because it contains a divergent subdiagram (the one on the left). Hence the name 'superficial' degree of divergence:

- a diagram with $D \geq 0$ can still be finite due to cancellations,
- a diagram with $D < 0$ can be divergent if it contains divergent subdiagrams,
- tree-level diagrams have $D = 0$ but they are finite.

Let's ignore these subtleties for a moment and assume that D counts the actual degree of divergence. From Eq. (8.66) we can determine it as

$$D = [\Gamma_n] - [\lambda]\, V\,. \tag{8.68}$$

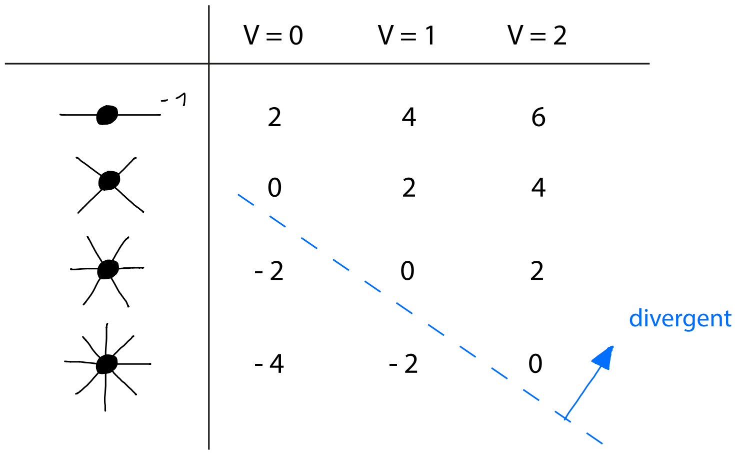
The mass dimension $[\Gamma_n]$ is fixed and does not depend on the order in perturbation theory, which is determined by V. However, D depends on V — it rises or falls with higher orders depending on the mass dimension of the coupling $[\lambda]$. Take ϕ^4 theory, where $[\lambda] = 0$ and D is independent of V:



Therefore, there are only two divergent Green functions in ϕ^4 theory: the inverse propagator and the four-point function. Those are exactly the ones with a tree-level term in the Lagrangian; they are also called the **primitively divergent Green functions**.

One can indeed show that the analysis goes through in general, also for divergent subdiagrams, which is known as the **BPHZ theorem** (Bogoliubov, Parasiuk, Hepp, Zimmermann). The reason is that the Z_i factors in front of the diagrams (which we can neglect at one-loop) cancel the divergences at higher orders. Take for example the two diagrams on the right in Eq. (8.67): both contribute to the six-point function, one with $V = 3$ and the other with $V = 4$. The $V = 3$ diagram carries factors $Z = 1 + \delta Z$, where δZ contributes at higher order to the $V = 4$ graph. The sum of all contributions at a given order cancels the divergences. Here it is especially useful to employ the counterterm language, because the subdivergences will cancel with the counterterms at each order in perturbation theory.

On the other hand, the same analysis for ϕ^6 theory gives us:



In other words, if we go high enough in perturbation theory eventually *every* Green function will diverge!

This leads to the notion of **renormalizability**: a theory is renormalizable if only a *finite* number of Green functions have $D \geq 0$, so that only a finite number of renormalization conditions are necessary to remove the divergences from the theory. From Eq. (8.68) this is equivalent to the following statement:

$$\text{A theory is renormalizable if} \quad [\lambda] \geq 0\,.$$

That is, the coupling must be either dimensionless or have a positive mass dimension (in the latter case the theory is called *super-renormalizable*). A non-renormalizable theory has a coupling with negative mass dimension: in that case every Green function eventually becomes divergent. Here we would need new renormalization conditions at each order in perturbation theory, and eventually infinitely many, so we must specify infinitely many constants from outside. The theory thereby loses its predictive power.

The good news is that we can read off a theory's renormalizability directly from its Lagrangian: we just need to look at the mass dimension of the coupling constant. For a scalar ϕ^p theory only ϕ^3 and ϕ^4 interactions are renormalizable whereas those with $p > 4$ are not. Renormalizability restricts the possible forms of interactions dramatically!