4.2 Spontaneous chiral symmetry breaking

In the quark model, the 'constituent-quark masses' enter as input parameters which cannot be further explained. How do they come about in QCD? This ties into the question of **mass generation**: if the light up and down quarks in the QCD Lagrangian have masses of a few MeV, how is it possible that the masses of the proton and other hadrons are of the order of 1 GeV? In fact, we could even set $m_u = m_d = 0$ and we would still get a proton mass not far from its physical value, so the overwhelming contribution to its mass must be generated in QCD.

Earlier we have seen that regularization introduces a scale. Without a scale in the theory, from a massless Lagrangian we would expect all hadrons to be massless as well, so the anomalous breaking of scale invariance is a necessary component. The other component is **spontaneous chiral symmetry breaking (SχSB)**. We will see that this mechanism plays a quite important role in the light hadron spectrum: it is not only responsible for the Goldstone nature of the pions, but also the origin of the constituent-quark masses which produce the typical hadronic scales of ∼1 GeV.

Spontaneous symmetry breaking. Let us go back to the beginning of Sec. (3.1) and start with some general considerations. Suppose ϕ_i are a set of (potentially composite) fields which transform nontrivially under some continuous global symmetry group G:

$$\phi_i' = D_{ij}(\varepsilon)\,\phi_j = \left(e^{i\sum_a \varepsilon_a \mathsf{t}_a}\right)_{ij}\phi_j = \phi_i + \delta\phi_i\,, \qquad \delta\phi_i = i\sum_a \varepsilon_a (\mathsf{t}_a)_{ij}\,\phi_j\,, \tag{4.2.1}$$

where ε_a are the group parameters, the t_a are the generators of the Lie algebra of G in the representation to which the ϕ_i belong, and $D(\varepsilon)$ are the representation matrices. The quantum-field theoretical version of this relation is

$$e^{i\sum_a \varepsilon_a Q_a}\,\phi_i\,e^{-i\sum_a \varepsilon_a Q_a} = D^{-1}_{ij}(\varepsilon)\,\phi_j\,. \tag{4.2.2}$$

where the charge operators Q_a form a representation of the algebra on the state space. Expanding the exponentials on both sides, we obtain for each ε_a:

$$[Q_a, \phi_i] = -(\mathsf{t}_a)_{ij}\,\phi_j\,. \tag{4.2.3}$$

We have encountered examples of this relation earlier:

- Eq. (3.1.71) for the quark field operators under a vector transformation;
- Eq. (3.1.68) for the collection of *composite* fields $\{S(x), S_a(x), P_a(x)\}$ under axial transformations (that this is a manifestation of the same relation will become clear in the discussion of the sigma model in Sec. 4.4.1).

If the symmetry group leaves the vacuum invariant, $e^{i\varepsilon_a Q_a}|0\rangle = |0\rangle$, then all generators Q_a must annihilate the vacuum: $Q_a|0\rangle = 0$. Hence, when we take the vacuum expectation value (VEV) of Eq. (4.2.2) we get

$$\langle 0\,|\phi_i|\,0\rangle = D^{-1}_{ij}(\varepsilon)\,\langle 0\,|\phi_j|\,0\rangle\,. \tag{4.2.4}$$

If the ϕ_i had been invariant under G to begin with, this relation would be trivially satisfied. Because they transform nontrivially, $D_{ij}^{-1}(\varepsilon)$ is not the identity matrix for all ε_a and so these vacuum expectation values must vanish:

$$Q_a\,|0\rangle = 0 \qquad \Rightarrow \qquad \langle 0\,|\phi_i|\,0\rangle = 0\,. \tag{4.2.5}$$

This is the '**Wigner-Weyl**' realization of a symmetry, which simply means that the symmetry is unbroken.

On the other hand, if an operator that is not invariant under G develops a nonzero vacuum expectation value $\langle 0\,|\phi_i|\,0\rangle \neq 0$, then the symmetry G is spontaneously broken. This is the '**Nambu-Goldstone** realization' of the symmetry, in which case we find

$$\langle 0|\,[Q_a, \phi_i]\,|0\rangle = -(\mathsf{t}_a)_{ij}\,\langle 0|\phi_j|0\rangle \neq 0\,. \tag{4.2.6}$$

Then we would conclude that the charges do not annihilate the vacuum: $Q_a|0\rangle \neq 0$. Since the symmetry is classically realized, they still commute with the Hamiltonian and we have found another energy-degenerate vacuum:

$$Q_a|0\rangle = |\eta\rangle \neq 0, \quad H|0\rangle = 0 \quad \Rightarrow \quad H|\eta\rangle = HQ_a|0\rangle = Q_aH|0\rangle = 0\,. \tag{4.2.7}$$

Unfortunately we have to be careful with these statements because in the case of spontaneous symmetry breaking the charges are not well defined. $|\eta\rangle$ is not a normalizable state, which we can see from using the definition of the charge (3.1.5) together with translation invariance:

$$\langle\eta|\eta\rangle = \langle 0|\,Q_a^2\,|0\rangle = \int d^3x \int d^3y\,\langle 0|\,j_a^0(x)\,j_a^0(y)\,|0\rangle = \infty\,. \tag{4.2.8}$$

Fortunately, *commutators* involving the charges are still well-defined, so when discussing spontaneous symmetry breaking we should start from Eq. (4.2.6). To prove the **Goldstone theorem**, we insert the completeness relation (2.2.5) in that equation and follow the same steps as when deriving the spectral representation:

$$\begin{aligned}
\langle 0|\,[Q_a(x_0), \phi(0)]\,|0\rangle &= \int d^3x\,\langle 0|\left[j_a^0(x), \phi(0)\right]|0\rangle \\
&= \sum_\lambda \int \frac{d^3p}{2E_p}\,\frac{i}{(2\pi)^3} \int d^3x\left(R_{a\lambda}(\boldsymbol{p})\,e^{-ipx} + R_{a\lambda}^\star(\boldsymbol{p})\,e^{ipx}\right) \\
&= \sum_\lambda \frac{i}{2m_\lambda}\left(R_{a\lambda}(\boldsymbol{0})\,e^{-im_\lambda x_0} + R_{a\lambda}^\star(\boldsymbol{0})\,e^{im_\lambda x_0}\right) \\
&= \sum_\lambda \frac{i}{m_\lambda}\,\mathrm{Re}\left\{R_{a\lambda}(\boldsymbol{0})\,e^{-im_\lambda x_0}\right\} \stackrel{!}{=} \text{const.}
\end{aligned} \tag{4.2.9}$$

In going from the first to the second row we used translation invariance (2.2.11) to factor out the phases $e^{\pm ipx}$, and we defined

$$\langle 0|\,j_a^0(0)\,|\lambda\rangle\langle\lambda|\,\phi(0)\,|0\rangle = iR_{a\lambda}(\boldsymbol{p})\,. \tag{4.2.10}$$

The integral over d^3x produces $\delta^3(\boldsymbol{p})$, so that $p_0 = E_p = (\boldsymbol{p}^2 + m_\lambda^2)^{1/2}$ becomes m_λ. By translation invariance, the VEV $\langle 0|\phi_j(x)|0\rangle = \langle 0|\phi_j(0)|0\rangle$ on the right-hand side of

Eq. (4.2.9) must also be independent of x_0, whereas the left-hand side still contains x_0 in the exponential. Thus, if the VEV is nonzero, the above requirement can only be met if for each charge Q_a there is a mode $|\lambda\rangle$ with

$$m_\lambda = 0 \qquad \text{and} \qquad \frac{R_{a\lambda}(\mathbf{0})}{m_\lambda} \neq 0\,. \tag{4.2.11}$$

Thus, for each generator that does not leave the vacuum invariant there is a **massless Goldstone boson**, which has a non-zero vacuum overlap $\langle 0|\, j_a^0(0)\,|\lambda\rangle$ and $\langle 0|\,\phi(0)\,|\lambda\rangle$. The other modes with $m_\lambda \neq 0$ (excited states) must have $R_{a\lambda}(\mathbf{0}) = 0$.

SχSB in QCD and chiral condensate. How does spontaneous breaking of chiral symmetry come about in QCD? The Goldstone theorem does not tell us *why* a non-zero VEV appears, it only says that *if* there is a non-zero VEV, we must have massless particles in the spectrum. Therefore, we must first identify potential candidates for vacuum condensates that break chiral symmetry. From Eq. (4.2.10) we already see that the 'field' $\phi(0)$ will have to be a composite operator, since only those produce overlaps with hadronic states.

Let us go back to the quark propagator,

$$S_{\alpha\beta}(x-y) = \langle 0|\mathsf{T}\,\psi_\alpha(x)\,\overline{\psi}_\beta(y)|0\rangle\,, \tag{4.2.12}$$

and contract it with either of the Dirac matrices $\Gamma \in \{\gamma^\mu,\, \gamma^\mu\gamma_5,\, \mathbb{1},\, i\gamma_5\}$ and flavor matrices $\{\mathsf{t}_a,\, 1\}$. This gives us the vacuum expectation values of either of the currents in Eq. (3.1.23):

$$-\,\Gamma_{\beta\alpha}\,\mathsf{t}_a\,S_{\alpha\beta}(0) = \langle 0|\, j_a^\Gamma(0)\,|0\rangle. \tag{4.2.13}$$

Because of translation invariance, they cannot depend on x and must be (dimensionful) constants. Due to Lorentz and parity invariance these must all be zero, with the only possible exception of the **scalar condensates** which carry the quantum numbers of the vacuum (0^{++}):

$$\begin{aligned} \langle 0|\,\tilde{S}_a(0)\,|0\rangle &= \langle 0|\,\overline{\psi}(0)\,\mathsf{t}_a\,\psi(0)\,|0\rangle\,, \\ \langle 0|\,\tilde{S}(0)\,|0\rangle &= \langle 0|\,\overline{\psi}(0)\psi(0)\,|0\rangle =: \langle\overline{\psi}\psi\rangle. \end{aligned} \tag{4.2.14}$$

Here we put a tilde on the scalar densities S and S_a to avoid confusion with the quark propagator. Actually, if $SU(N_f)$ were unbroken, all flavor non-singlet scalar condensates would vanish as well. From Eq. (3.1.57) one can derive

$$[Q_a^V, \tilde{S}_b(x)] = i f_{abc}\,\tilde{S}_c(x)\,, \tag{4.2.15}$$

and since unbroken $SU(N_f)_V$ implies $Q_a^V|0\rangle = 0$, the VEV of this relation vanishes. The singlet condensate is then identical for all flavors:

$$\langle 0|\tilde{S}_a(0)|0\rangle = 0 \quad \Rightarrow \quad \langle\bar{u}u\rangle - \langle\bar{d}d\rangle = 0, \quad \langle\bar{u}u\rangle + \langle\bar{d}d\rangle - 2\langle\bar{s}s\rangle = 0, \tag{4.2.16}$$

and therefore $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = \langle\bar{s}s\rangle = \langle\overline{\psi}\psi\rangle/3$.

Finally, in the discussion below Eq. (3.1.47) we saw that a scalar bilinear of quarks breaks chiral symmetry, i.e., it breaks $SU(N_f)_A \times U(1)_A$. Thus we have a potential candidate for a condensate that breaks chiral symmetry. In a chirally symmetric theory of massless quarks, this quantity should vanish — but does it?

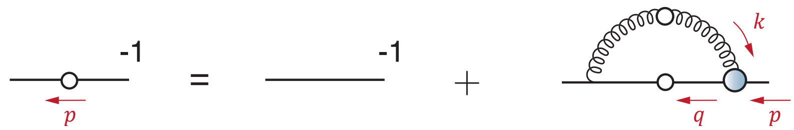

FIG. 4.3: Quark DSE.

Quark mass function. Since the quark condensate is the trace of the quark propagator, let us have a closer look at the propagator itself. For the following discussion we temporarily switch to **Euclidean conventions** to avoid cumbersome factors of $i\epsilon$. The transcription rules between Minkowski and Euclidean space can be found in Appendix C, but all we need to remember in the following is $p^2 = -p_E^2$ and the quark propagator in Euclidean conventions (we drop the subscript E):

$$S(p) = \frac{1}{A(p^2)} \frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2} \,. \tag{4.2.17}$$

Recall the quark Dyson-Schwinger equation (DSE) in Fig. 4.3,

$$S^{-1}(p) = A(p^2)\,\big(i\not{p} + M(p^2)\big) = Z_2(i\not{p} + Z_m m) + \Sigma(p)\,, \tag{4.2.18}$$

where $M(p^2)$ is the **quark mass function** and $\Sigma(p)$ the self-energy incorporating the quantum effects, which in one-loop perturbation theory reduces to Eq. (2.3.46). To obtain the quark condensate for a particular flavor, we need to take the Dirac and color trace of the quark propagator, which singles out the term with $M(p^2)$ and gives a factor $4N_c$. In addition, setting $x - y = 0$ corresponds to an integration over $d^4p/(2\pi)^4$ in momentum space, which from Eq. (C.32) entails

$$\int \frac{d^4p}{(2\pi)^4}\, f(p^2) = \frac{1}{(4\pi)^2} \int dp^2\, p^2\, f(p^2)\,. \tag{4.2.19}$$

Thus we arrive at[2]

$$-\langle \bar{u}u\rangle = N_c \int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\, S(p) = \frac{N_c}{(2\pi)^2} \int dp^2 \frac{p^2}{A(p^2)} \frac{M(p^2)}{p^2 + M(p^2)^2}\,. \tag{4.2.20}$$

The functions $M(p^2)$ and $A(p^2)$ should be positive for spacelike momenta $p^2 \geq 0$. Since for a chirally symmetric Lagrangian ($m = 0$) we expect the condensate to vanish, and because the condensate is proportional to the integrated quark mass function, this means that the quark mass function should be zero for all p^2. The resulting quark propagator is then chirally symmetric: $\{\gamma_5, S(p)\} = 0$.

Indeed, this is what happens when we evaluate the self-energy order by order in perturbation theory, see Fig. 4.4. In the massless theory, the tree-level propagator is proportional to $\not{p}$ and the tree-level vertex is proportional to γ^μ, so they both contain one γ matrix. However, *every* possible perturbative diagram has an odd number of γ matrices whose Dirac trace vanishes. In this way, we can never generate a mass function and $M(p^2) = 0$ to all orders in perturbation theory!

[2] Strictly speaking we should also attach a factor $Z_2 Z_m$, since the condensate renormalizes like the mass term in the Lagrangian and the product $m\langle\bar{\psi}\psi\rangle$ is renormalization-point independent.

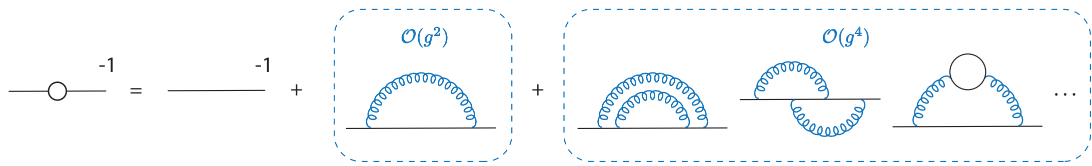

FIG. 4.4: Perturbative expansion of the inverse quark propagator. In massless QCD, each Feynman diagram contains an odd number of γ matrices whose trace vanishes.

On the other hand, we can generate a non-zero mass function **nonperturbatively**, which can already be illustrated in simple DSE models. Apart from renormalization constants, the exact expression for the self-energy is

$$\Sigma(p) = g^2 C_F \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu\, S(q)\, \Gamma^\nu(q,p)\, D^{\mu\nu}(k)\,, \tag{4.2.21}$$

which depends on the full gluon propagator and quark-gluon vertex. Let us assume that the quark-gluon vertex remains at tree-level, so that only the internal quark and gluon propagators are dressed ('rainbow truncation'). In Feynman gauge the gluon is diagonal in its Lorentz indices, so we can write the self-energy as

$$\Sigma(p) = \int d^4k\, \gamma^\mu\, S(p+k)\, \gamma^\mu\, D(k)\,, \tag{4.2.22}$$

where $D(k)$ is proportional to the gluon propagator and absorbs all prefactors. Thus, if we can find a good ansatz for $D(k)$, we can solve the Dyson-Schwinger equation $S^{-1}(p) = i\not{p} + m + \Sigma(p)$ for the quark propagator. $D(k)$ must be a scalar function of the gluon momentum k^2 with mass dimension -2. At large k^2 it should be proportional to QCD's running coupling, $D(k^2) \propto \alpha_s(k^2)/k^2$, because this is where quarks and gluons become asymptotically free. In the following we employ two rather crude models: one where the gluon propagator is localized in momentum space and another one where it is localized in coordinate space.

Munczek-Nemirovsky model. In this case the gluon propagator is just a δ-function peaked at the origin, equipped with some mass scale Λ:

$$D(k) = \Lambda^2\, \delta^4(k)\,. \tag{4.2.23}$$

Here the self-energy can be integrated analytically, so the model is UV-finite and instead of imposing renormalization conditions we can set all renormalization constants to 1 (as we already did above). The result is

$$\Sigma(p) = \Lambda^2\, \gamma^\mu S(p) \gamma^\mu = \Lambda^2\, \frac{\gamma^\mu\,(-i\not{p} + M)\,\gamma^\mu}{(p^2+M^2)\,A} = 2\Lambda^2\, \frac{i\not{p} + 2M}{(p^2+M^2)\,A}\,, \tag{4.2.24}$$

where we suppressed the momentum dependencies of $A(p^2)$ and $M(p^2)$ to avoid clutter. Putting this back into the DSE leads to selfconsistent algebraic equations for the two quark dressing functions:

$$A = 1 + \frac{2\Lambda^2}{(p^2+M^2)\,A}\,, \qquad AM = m + 2M\, \frac{2\Lambda^2}{(p^2+M^2)\,A}\,. \tag{4.2.25}$$

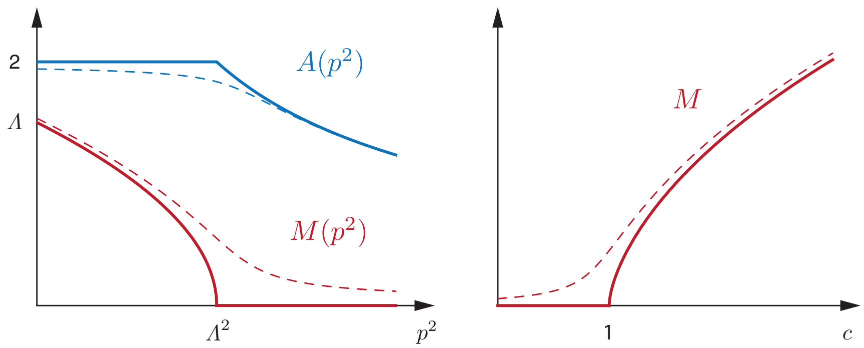

FIG. 4.5: Quark propagator in the Munczek-Nemirovsky model (left) and NJL model (right). The solid lines are the results in the chiral limit and the dashed lines exemplify the solutions for $m \neq 0$.

In the chiral limit ($m = 0$), we see from the second equation that the trivial solution $M = 0$ is always possible. It leads to a quadratic equation for A whose result is

$$M(p^2) = 0\,, \qquad A(p^2) = \tfrac{1}{2}\left(1 + \sqrt{1 + 8\,\Lambda^2/p^2}\right). \tag{4.2.26}$$

It has the correct perturbative behavior for $p^2 \to \infty$, namely $M = 0$ and $A \to 1$, so it reverts the quark propagator back to its tree-level form and preserves chiral symmetry. On the other hand, $A(p^2)$ diverges for $p^2 \to 0$, so this cannot be the whole story. Indeed there is another solution with $M \neq 0$:

$$M(p^2) = \sqrt{\Lambda^2 - p^2}\,, \qquad A(p^2) = 2\,. \tag{4.2.27}$$

It breaks chiral symmetry and is finite in the infrared. Both solutions are connected at the point $p^2 = \Lambda^2$, see Fig. 4.5. This is the typical shape of an order parameter of a spontaneously broken symmetry, like the magnetization in a ferromagnet when plotted over temperature. If we switch on a quark mass $m \neq 0$, the curves become smooth (in the ferromagnet this corresponds to a background magnetic field).

Despite the simplicity of the model, these results already capture the essence of more realistic DSE calculations. At large momenta, $M(p^2)$ is the renormalized current-quark mass in the Lagrangian. When lowering the momentum, the onset of the non-symmetric phase sets in at some typical hadronic scale Λ, below which a mass is spontaneously generated. The mass function in the infrared defines the quark mass at low momenta that is relevant for hadrons, so it can be viewed as a 'constituent-quark' mass scale. Thus, the quark mass function encodes the transition from a current quark at large momenta to a constituent quark in the infrared, and this effect cannot be described in QCD perturbation theory.

If we insert the combined solution in Eq. (4.2.20), the resulting quark condensate in the chiral limit becomes

$$-\langle \bar{u}u\rangle = \frac{N_c}{(2\pi)^2}\int_0^{\Lambda^2} dp^2\, p^2\, \frac{\sqrt{\Lambda^2 - p^2}}{2\Lambda^2} = \frac{2}{15}\,\frac{N_c}{(2\pi)^2}\,\Lambda^3\,. \tag{4.2.28}$$

With $\Lambda = 1$ GeV we even get a reasonable numerical value: $-\langle \bar{u}u\rangle \sim (220\,\text{MeV})^3$.

NJL model/contact interaction. The shortcoming of the Munczek-Nemirovsky model is that it does not have a *critical* coupling: a non-trivial solution for the quark mass function and thus a chiral condensate exist for any $\Lambda > 0$. The gluon propagator in Eq. (4.2.23) is localized in momentum space because of the $\delta-$function. We could take the extreme opposite and localize it in coordinate space, which results in an effective four-fermi contact interaction between two quarks where the gluon shrinks to a point and is integrated out. This is the **NJL model** (Nambu, Jona-Lasinio), where the momentum dependence of the gluon is simply a constant:

$$D(k) = \frac{1}{(2\pi)^2}\,\frac{c}{\Lambda^2}\,. \tag{4.2.29}$$

In this case it is more convenient to integrate over the quark momentum $q = p - k$ instead of k in (4.2.22). However, now the self-energy integral must be regulated because it is divergent. We could impose a sharp cutoff at $q^2 = \Lambda^2$, so that the gluon propagator is a constant up to some scale Λ and vanishes above. As a consequence, the integrand no longer depends on the external momentum p,

$$\Sigma(p) = \frac{1}{(2\pi)^2}\,\frac{c}{\Lambda^2}\int d^4q\,\gamma^\mu\,S(q)\,\gamma^\mu = \frac{1}{(2\pi)^2}\,\frac{c}{\Lambda^2}\int d^4q\,\frac{2}{A(q^2)}\,\frac{i\not{q} + 2M(q^2)}{q^2 + M(q^2)^2}\,, \tag{4.2.30}$$

which means that $\Sigma(p)$ is constant and therefore A and M will be constants as well. The integral over $\not{q}$, which is the self-energy contribution to A, vanishes and we get $A = 1$. The equation for M becomes:

$$M = m + cM\int_0^1 dy\,\frac{y}{y+a} = m + cM\left[1 - a\,\ln(1+\tfrac{1}{a})\right] = m + cM\,f(a)\,, \tag{4.2.31}$$

where we set $y = q^2/\Lambda^2$ and $a = M^2/\Lambda^2$. The function $f(a)$ satisfies $f(a) \leq 1$ and $f(0) = 1$. In the chiral limit we obtain the algebraic equation

$$M = cM\,f(a)\,, \tag{4.2.32}$$

which returns again the trivial solution $M = 0$, but also a nontrivial solution where M as a function of c is determined from the equation $f(a) = 1/c$. Because $f(a) \leq 1$, this solution only occurs above a critical value $c \geq 1$.

The result is shown in Fig. 4.5: In contrast to the previous case, the dynamical quark mass M is no longer a mass *function* that depends on the momentum but just a constant; however, it depends on the coupling strength c and vanishes for $c < 1$. Above that value, chiral symmetry is spontaneously broken. If we plug the result into the chiral condensate (4.2.20) using the same cutoff, we obtain the same form as in Eq. (4.2.28) except that the prefactor $2/15$ is replaced by $M(c)/(c\Lambda)$, which also vanishes for $c < 1$.

In general, the gluon propagator is neither a $\delta-$function nor a constant, and the spontaneous breaking of chiral symmetry will not only generate a mass term for the quark propagator but also chirally asymmetric terms for other correlation functions with quark and antiquark legs such as the quark-gluon vertex. Nevertheless, both models encode general features:

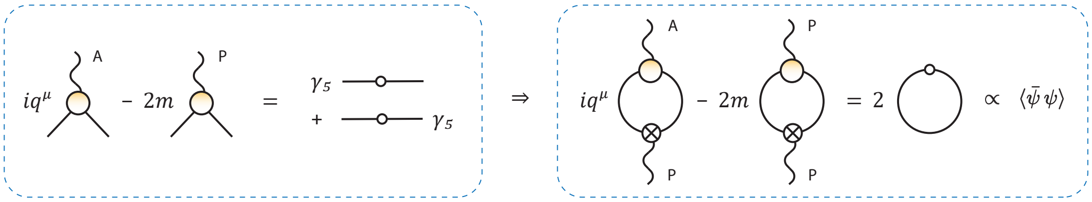

Fig. 4.6: Axialvector WTI for the three-point functions (left) and current correlators (right).

- Implementing a scale Λ was necessary to make them work. If we replace Λ^2 by k^2 in Eq. (4.2.23), the self-energy vanishes. In the NJL model, Λ is the regulator which cannot be removed. The quark mass function and other dimensionful quantities such as the chiral condensate, and eventually the masses of hadrons, are then proportional to this scale, so that SχSB can be viewed as the **mass generation** mechanism in the fermion sector of QCD.
- SχSB is a *critical* phenomenon: if the combined strength from the gluon propagator and quark-gluon vertex (the 'effective' running coupling) exceeds a critical value, a quark mass is generated dynamically; otherwise we remain with the chirally symmetric solution.
- In contrast to effective theories of QCD, where the terms that trigger SχSB already appear in the Lagrangian, the QCD Lagrangian tells us nothing about whether chiral symmetry is preserved at the quantum level or not. Its spontaneous breaking is a purely dynamical effect induced by the strong gluonic interactions, hence the name **dynamical chiral symmetry breaking (DCSB)**.

Gell-Mann-Oakes-Renner relation. Now let us return to the Goldstone theorem. We have explored the origin of SχSB and identified its order parameters: the scalar quark condensate or, equivalently, the quark mass function. Hence, any other quantity that depends on the mass function (and vanishes if the mass function does) will break chiral symmetry as well. In Eq. (3.1.143) we found that, as a simple consequence of the PCAC relation, either a pseudoscalar meson's mass or its electroweak decay constant must vanish in the chiral limit:

$$f_\lambda\, m_\lambda^2 = 2m\, r_\lambda \quad \xrightarrow{m=0} \quad 0\,. \tag{4.2.33}$$

Therefore, if we can show that the pion decay constant f_π is also proportional to the mass function and comes about by SχSB, we must have massless pions.

The right place to look for such a relation is the axialvector WTI in (3.1.81), which is pictorially shown in Fig. 4.6. On its l.h.s. we have the difference of the G_A and G_P three-point functions; the r.h.s. is the sum of quark propagators multiplied with γ_5. If we multiply again with γ_5 and take the trace, we get a difference of AP and PP current correlators on the left and the quark condensate on the right. When inserting the completeness relation, both terms contain pseudoscalar poles *only*, where the residues depend on f_λ and r_λ as given in Eq. (3.1.144). Moreover, the hadronic poles must cancel out between G_A and G_P because the quark propagator does not have such poles. In this way we should be able to establish a relation between f_π and $\langle\bar{\psi}\psi\rangle$.

Let us start directly from the WTI (3.1.72) for the AP current correlator:

$$\partial_\mu^x \langle 0|\mathsf{T}\, A_a^\mu(x)\, P_b(0)|0\rangle - 2m\,\langle 0|\mathsf{T}\, P_a(x)\, P_b(0)|0\rangle = \delta(x^0)\,\langle 0|\left[A_a^0(x), P_b(0)\right]|0\rangle\,. \quad (4.2.34)$$

We already inserted the PCAC relation for the PP term. If we integrate over d^4x on the r.h.s., we obtain the vacuum expectation value of the commutator that we derived earlier in Eq. (3.1.68),

$$\langle 0|\left[Q_a^A, P_b(0)\right]|0\rangle = -i\langle 0|\left[\frac{\delta_{ab}}{N_f}\,S(0) + d_{abc}\,S_c(0)\right]|0\rangle = -i\,\frac{\delta_{ab}}{N_f}\,\langle\bar\psi\psi\rangle\,, \qquad (4.2.35)$$

where only the singlet condensate survives in the limit of exact $SU(N_f)_V$. This is the representative of the generic equation (4.2.6): since the condensate which is not invariant under axial symmetries is the scalar condensate and the respective charges are the axial charges, the corresponding field φ_i must be the pseudoscalar density. For the l.h.s. in Eq. (4.2.34), we insert the spectral decompositions of the AP and PP current correlators from (3.1.144) and (3.1.145) and integrate over x. This means taking the limit $p \to 0$:

$$\lim_{p\to 0}\sum_\lambda \frac{p^2 f_\lambda - 2m\, r_\lambda}{p^2 - m_\lambda^2 + i\varepsilon}\, ir_\lambda\,\delta_{ab} = \sum_\lambda ir_\lambda f_\lambda\,\delta_{ab} \stackrel{!}{=} -i\,\frac{\delta_{ab}}{N_f}\,\langle\bar\psi\psi\rangle\,, \qquad (4.2.36)$$

where we used the relation $f_\lambda\, m_\lambda^2 = 2mr_\lambda$ in the second equality. The poles cancel indeed, and we arrive at the result that if chiral symmetry is realized and the quark condensate vanishes, all combinations $r_\lambda\, f_\lambda$ must vanish as well; if it is spontaneously broken, there is at least one mode where both r_λ and f_λ are nonzero. Since $f_\lambda \neq 0$ in that case, we must have $m_\lambda \to 0$, i.e., a massless Goldstone boson.

Each $|\lambda\rangle$ corresponds to one of the generators, so there is a massless Goldstone boson for each generator t_a (for three flavors with $SU(3)_A \times U(1)_A$ this means a pseudoscalar octet and a singlet). In turn, the decay constants f_λ must vanish for the remaining excited states with $m_\lambda \neq 0$, so we can remove the sum in the equation above and write

$$r_{\lambda_0}\, f_{\lambda_0} = -\frac{\langle\bar\psi\psi\rangle}{N_f}\,, \qquad (4.2.37)$$

where $|\lambda_0\rangle$ is the ground state in each channel. If we substitute r_{λ_0} by the condensate and insert it in Eq. (4.2.33), we obtain the **Gell-Mann-Oakes-Renner (GMOR)** relation,

$$f_{\lambda_0}^2\, m_{\lambda_0}^2 = -2m\,\frac{\langle\bar\psi\psi\rangle}{N_f}\,, \qquad (4.2.38)$$

which is valid for each member of the lowest-lying pseudoscalar octet and singlet. (In the singlet case it only holds if we ignore the anomaly.)

All in all, SχSB has important consequences for the light hadron spectrum: It generates a large dynamical quark mass function, which translates to a large mass contribution for hadrons made of quarks and antiquarks even in the chiral limit. The pseudoscalar meson masses, on the other hand, behave like $m_{\text{PS}}^2 \propto m_q$ and vanish for $m_q \to 0$ as shown in Fig. 4.7.

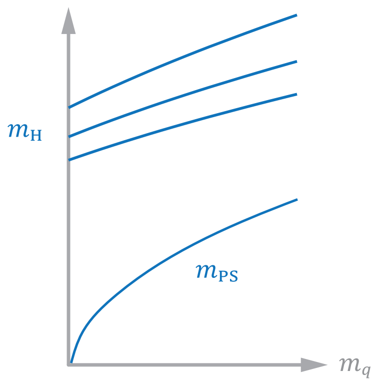

FIG. 4.7: Generic dependence of hadron masses on the current-quark mass

$SU(N_f)_V$ breaking. So far we have assumed that all quark masses are equal, $m_u = m_d = m_s$. In the case of $SU(3)_V$ breaking, we have to go back to the general PCAC relation (3.1.38) and evaluate the anticommutators, and also keep the d_{abc} terms in Eq. (4.2.35). In this case the form of the GMOR relation remains the same for each generator with index a if we replace the quark mass m by

$$\begin{aligned} a = 1,2,3: &\quad \tfrac{1}{2}\,(m_u + m_d)\,, \\ a = 4,5: &\quad \tfrac{1}{2}\,(m_u + m_s)\,, \\ a = 6,7: &\quad \tfrac{1}{2}\,(m_d + m_s)\,, \\ a = 8: &\quad \tfrac{1}{6}\,(m_u + m_d + 4m_s)\,, \\ a = 0: &\quad \tfrac{1}{3}\,(m_u + m_d + m_s)\,, \end{aligned} \tag{4.2.39}$$

and the condensate accordingly:

$$\frac{\langle\bar{\psi}\psi\rangle}{3} \longrightarrow \frac{\langle\bar{u}u + \bar{d}d\rangle}{2} \quad (a=1,2,3), \qquad \frac{\langle\bar{u}u + \bar{s}s\rangle}{2} \quad (a=4,5), \quad \text{etc.} \tag{4.2.40}$$

Then we get for the pions and kaons:

$$f_\pi^2\, m_\pi^2 = -\frac{m_u + m_d}{2}\,\langle\bar{u}u + \bar{d}d\rangle\,, \qquad f_K^2\, m_K^2 = -\frac{m_u + m_s}{2}\,\langle\bar{u}u + \bar{s}s\rangle\,. \tag{4.2.41}$$

Inserting the experimental values[3] $f_\pi \approx 92$ MeV, $m_\pi \approx 140$ MeV and assuming an average quark mass $m_u = m_d = 3.5$ MeV yields $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle \approx -(280\,\text{MeV})^3$. The same estimate for kaons ($f_K \approx 110$ MeV, $m_K \approx 494$ MeV, $m_s \approx 120$ MeV) gives us $\langle\bar{s}s\rangle \approx -(290\,\text{MeV})^3$. The renormalized quark masses and condensates are renormalization-point and -scheme dependent; the values quoted here are consistent with lattice QCD results[4] obtained in an $\overline{\text{MS}}$ scheme at $\mu = 2$ GeV.

Strictly speaking, the GMOR relation as it stands is only valid in the chiral limit because the quark condensate is only well-defined for $m = 0$. We can see this from its definition (4.2.20) as the momentum integral of the quark mass function: In the chiral limit, $M(p^2 \to \infty)$ vanishes like $1/p^2$, so the integral only diverges logarithmically and is renormalized by $Z_2 Z_m$. For $m \neq 0$, the one-loop result in Eq. (2.3.88)) entails that the mass function vanishes logarithmically and therefore the integral diverges quadratically. In this case, $f_\lambda\, m_\lambda^2 = 2mr_\lambda$ can be viewed as a generalized GMOR relation since the quantities f_λ and r_λ are well-defined for all quark masses. In principle, they can be used to define the quark condensate from a pseudoscalar meson's Bethe-Salpeter wave function, namely as the chiral limit of the combination $r_{\lambda_0} f_{\lambda_0}$ via Eq. (4.2.37).

[3] The decay constants are sometimes defined with a factor $\sqrt{2}$, in which case $f_\pi \approx 130$ MeV.
[4] McNeile et al., Phys. Rev. D87 (2013), 034503. arXiv:1211.6577.