Chapter 2

QCD

Quantum Chromodynamics (QCD) is the theory of the strong interaction. It describes the 'color force' that binds quarks and gluons to colorless hadrons (protons, neutrons, pions, etc.) and hadrons to nuclei. At hadronic scales, the strong force is ~ 100 times stronger than the electromagnetic interaction, extremely short-ranged (the typical interaction range is the size of a hadron $\sim 1\,\text{fm} = 10^{-15}\,\text{m}$), and its typical energy scale is the mass of the proton ~ 1 GeV.

The strong interaction is described by a local, non-Abelian $SU(3)_C$ gauge theory with several peculiar features. While quarks and gluons are asymptotically free at short distances, they are confined at large distances: only colorless bound states (hadrons) can be detected in experiments, and no quark or gluon has ever been observed directly. Nevertheless, nature has given us an abundance of evidence that these constituents exist, and their theoretical description in terms of a non-Abelian gauge theory has evolved from being considered a mere mathematical trick to a quite fundamental framework. In this chapter we will recapitulate the properties of QCD and its fundamental degrees of freedom and postpone the discussion of hadrons to Chapter 3.

2.1 QCD Lagrangian

Field content. The definition of a quantum field theory starts with constructing its Lagrangian $\mathcal{L}$ (or, equivalently, its action $S = \int d^4x\, \mathcal{L}$), based on the desired underlying symmetries. The symmetries of QCD are: Poincaré invariance, local color gauge invariance and various flavor symmetries, and the fields in the Lagrangian should transform under representations of these groups. The QCD Lagrangian contains quark and antiquark fields, and (as a consequence of color gauge invariance) gluon fields which mediate the strong interaction:

$$\psi_{\alpha,i,f}(x)\,, \qquad \overline{\psi}_{\alpha,i,f}(x)\,, \qquad A_a^\mu(x)\,. \tag{2.1.1}$$

The quark fields are Dirac spinors (index α) and transform under the fundamental representation of $SU(3)_C$ (color index $i = 1, 2, 3$ or red, green blue). The additional index $f = 1 \dots N_f$ labels the flavor quantum number ($f =$ up, down, strange, charm, bottom, top). The eight gluon fields $A_a^\mu(x)$ are Lorentz vectors; there is one field for each generator $\mathtt{t}_a$ of the group ($a = 1 \dots 8$). In the fundamental representation: $\mathtt{t}_a = \lambda_a/2$, where the λ_a are the eight Gell-Mann matrices; see Appendix A for a collection of basic $SU(N)$ relations. Gluons are flavor-blind and carry no flavor labels.

Gauge invariance. A free fermion Lagrangian $\bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi$ constructed from the quark and antiquark fields (we leave the summation over Dirac, color and flavor indices implicit) is invariant under global $SU(3)_C$ transformations

$$\psi'(x) = U\psi(x), \qquad \bar{\psi}'(x) = \bar{\psi}(x)\,U^\dagger \qquad \text{with} \quad U = e^{i\varepsilon} = e^{i\sum_a \varepsilon_a \mathtt{t}_a}\,, \tag{2.1.2}$$

where $\varepsilon_a = const.$ and the U_{ij} act on the color indices of the quarks. This invariance is no longer satisfied if we impose a local $SU(3)_C$ gauge symmetry $\psi'(x) = U(x)\,\psi(x)$ with spacetime-dependent group parameters $\varepsilon_a(x)$. The mass term is still invariant, but the derivative in the kinetic term now also acts on the spacetime argument of $U(x)$, and invariance of the Lagrangian (or the action) cannot be satisfied with an ordinary partial derivative. To ensure local color gauge invariance, we introduce a **covariant derivative** and thus gluon fields:

$$D_\mu = \partial_\mu - igA_\mu\,, \tag{2.1.3}$$

where $A^\mu(x) = \sum A^\mu_a(x)\,\mathtt{t}_a$ is an element of the Lie algebra. From the new Lagrangian $\bar{\psi}\,(iD\!\!\!/ - m)\,\psi$ we see that $D\!\!\!/\,\psi$ must transform in the same way as the quark field itself, which fixes the transformation properties of the gluon fields:

$$\bar{\psi}'\,D\!\!\!/'\psi' \stackrel{!}{=} \bar{\psi}\,D\!\!\!/\,\psi \quad \Rightarrow \quad D'_\mu\,\psi' = UD_\mu\psi = UD_\mu U^\dagger\psi' \tag{2.1.4}$$

$$\Rightarrow \quad (\partial_\mu - igA'_\mu)\,\psi' = U\,(\partial_\mu - igA_\mu)\,U^\dagger\psi'$$

$$\Rightarrow \quad A'_\mu = UA_\mu U^\dagger + \frac{i}{g}\,U(\partial_\mu U^\dagger). \tag{2.1.5}$$

The second term in A'_μ is particular to local gauge transformations; for a global symmetry we don't need a covariant derivative and could simply set $A_\mu = 0$. Note also that we can generate gluon fields out of nothing ($A_\mu = 0$) by a local gauge transformation: such gauge fields $\sim U(\partial_\mu U^\dagger)$ are called pure gauge configurations.

Why do we actually impose local gauge invariance in the first place? In fact, only *global* symmetries are true 'symmetries' which lead to conserved charges and quantum numbers. A local gauge symmetry reflects a redundancy in the description, which can be seen if we turn the argument around and start from Eq. (2.1.5), for example in the Abelian case where $U(x) = e^{i\varepsilon(x)}$ is just a phase. The action of a free massless vector field contains redundant degrees of freedom which are related to each other by local gauge transformations $A'_\mu = A_\mu + \partial_\mu\varepsilon/g$. The standard way to eliminate them is to modify the Lagrangian and impose a gauge-fixing condition on the state space (cf. Sec. 2.2.3). As a consequence, longitudinal and timelike photons decouple from physical processes and S-matrix elements are transverse: $q_\mu\mathcal{M}^\mu = 0$. To preserve this feature when including interactions (e.g., when adding fermions), the interacting part of the action must couple to a conserved current corresponding to the global symmetry of the full action, $\delta S_{\text{int}}/\delta A^\mu = j^\mu$, which is equivalent to imposing *local* gauge invariance for the matter fields. Thus, Eq. (2.1.5) is tied to the invariance under $\psi'(x) = U(x)\,\psi(x)$, and even though we needed an underlying global symmetry in the fermion sector to begin with, the local gauge invariance is not truly a symmetry but rather a *consistency* constraint that generates dynamics. In QCD, it introduces a quark-gluon interaction of the form $g\,\bar{\psi}A\!\!\!/\,\psi$.

Another way to motivate the covariant derivative is the following. We can write the ordinary derivative as

$$n^\mu \partial_\mu \psi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - \psi(x)\right]. \tag{2.1.6}$$

For a local gauge transformation $\psi'(x) = U(x)\psi(x)$ the first term becomes $U(x+\epsilon n)\psi(x+\epsilon n)$ but the second $U(x)\psi(x)$, so we are comparing objects at different spacetime points. To remedy this, we define the **parallel transporter** or **link variable** $C(y,x)$ by

$$C'(y,x) = U(y)\, C(y,x)\, U^\dagger(x)\,, \qquad C(x,x) = 1\,, \tag{2.1.7}$$

because then the quantity $C(y,x)\,\psi(x)$ has a simple transformation behavior:

$$[C(y,x)\,\psi(x)]' = U(y)\,C(y,x)\,U^\dagger(x)\,U(x)\,\psi(x) = U(y)\,C(y,x)\,\psi(x)\,. \tag{2.1.8}$$

Now, if we define the covariant derivative as

$$n^\mu D_\mu \psi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - C(x+\epsilon n, x)\,\psi(x)\right] \tag{2.1.9}$$

and perform a gauge transformation, then $U(x+\epsilon n)$ can be pulled out so that $[D_\mu\psi(x)]' = U(x) D_\mu \psi(x)$, and thus $\overline{\psi}\, \not{D}\, \psi$ is invariant under the local symmetry.

Moreover, we can write down the Taylor expansion of the parallel transporter:

$$C(x+\epsilon n, x) = 1 + \epsilon n^\mu i g A_\mu(x) + \mathcal{O}(\epsilon^2)\,, \tag{2.1.10}$$

where igA_μ is just a *name* for the coefficient of the linear term. Inserting this into (2.1.9) yields

$$n^\mu D_\mu \psi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - \psi(x) - \epsilon n^\mu i g A_\mu\, \psi(x)\right] = n^\mu \partial_\mu \psi(x) - n^\mu i g A_\mu \psi(x) \tag{2.1.11}$$

and therefore $D_\mu = \partial_\mu - igA_\mu$. Similarly, the transformation of the gluon field follows from

$$\begin{aligned} C'(x+\epsilon n, x) &= U(x+\epsilon n)\, C(x+\epsilon n, x)\, U^\dagger(x) \\ &= \left[U(x) + \epsilon n^\mu \partial_\mu U(x) + \mathcal{O}(\epsilon^2)\right]\left[1 + \epsilon n^\nu i g A_\nu(x) + \mathcal{O}(\epsilon^2)\right] U^\dagger(x) \\ &= 1 + \epsilon n^\mu i g \left[U(x) A_\mu(x)\, U^\dagger(x) - \frac{i}{g}\left(\partial_\mu U(x)\right) U^\dagger(x)\right] + \mathcal{O}(\epsilon^2) \\ &\overset{!}{=} 1 + \epsilon n^\mu i g A'_\mu(x) + \mathcal{O}(\varepsilon^2)\,, \end{aligned} \tag{2.1.12}$$

which reproduces the result (2.1.5) since $\partial_\mu(UU^\dagger) = (\partial_\mu U)\, U^\dagger + U \partial_\mu U^\dagger = 0$.

Gluon dynamics. Next, we need a kinetic term that describes the dynamics of the gluons. To this end we define the **gluon field strength tensor** as the commutator of two covariant derivatives:

$$F_{\mu\nu}(x) = \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]. \tag{2.1.13}$$

It is then also an element of the Lie algebra and we can write it as

$$F_{\mu\nu} = \sum_a F^a_{\mu\nu}\, \mathsf{t}_a\,, \tag{2.1.14}$$

where the t_a are again taken in the fundamental representation because ∂_μ, D_μ and A_μ act on quark fields in the (three-dimensional) fundamental representation of $SU(3)_C$. $F_{\mu\nu}$ inherits the transformation properties from (2.1.4): $F'_{\mu\nu} = U F_{\mu\nu} U^\dagger$.

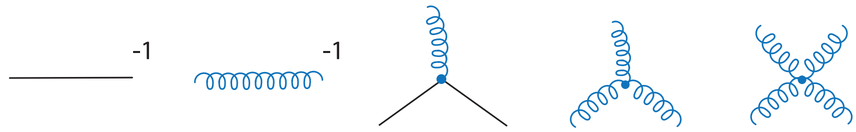

Fig. 2.1: Tree-level (inverse) propagators and interactions in the QCD action.

The contraction of two field-strength tensors is not gauge invariant; only its color trace is invariant due to the cyclic property of the trace:

$$\mathrm{Tr}\left\{F'_{\mu\nu}F'^{\mu\nu}\right\} = \mathrm{Tr}\left\{UF_{\mu\nu}U^\dagger\, UF^{\mu\nu}U^\dagger\right\} = \mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}. \tag{2.1.15}$$

Only the trace can therefore appear in the Lagrangian. We can write it as

$$\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} = F^a_{\mu\nu}\,F^{\mu\nu}_b\,\mathrm{Tr}\left\{\mathsf{t}_a\,\mathsf{t}_b\right\} = T(R)\,F^a_{\mu\nu}\,F^{\mu\nu}_a\,, \tag{2.1.16}$$

where $T(R) = 1/2$ in the fundamental representation of $SU(N)$, cf. Appendix A. From Eq. (2.1.5) we also conclude that a gluon mass term $\sim m_g\,A_\mu A^\mu$ cannot appear in the Lagrangian because it would violate gauge invariance: gluons must be **massless**.

We can work out the components of the field-strength tensor as

$$\begin{aligned} F_{\mu\nu} = F^a_{\mu\nu}\,\mathsf{t}_a &= \partial_\mu A^a_\nu\,\mathsf{t}_a - \partial_\nu A^a_\mu\,\mathsf{t}_a - ig\,A^a_\mu A^b_\nu\,[\mathsf{t}_a,\mathsf{t}_b] \\ &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc}\,A^b_\mu\,A^c_\nu\right)\mathsf{t}_a\,, \end{aligned} \tag{2.1.17}$$

where we used $[\mathsf{t}_a,\mathsf{t}_b] = if_{abc}\,\mathsf{t}_c$. Note that in an Abelian gauge theory such as QED this commutator would vanish, leaving only the linear terms in the gluon fields. The non-Abelian nature of $SU(3)_C$ introduces gluonic self-interactions which lead to significant complications. Inserting Eq. (2.1.17) into the term $F^a_{\mu\nu}\,F^{\mu\nu}_a$ and partial integration yields

$$\begin{aligned} -\tfrac{1}{4}F^a_{\mu\nu}\,F^{\mu\nu}_a \;\cong\; &\frac{1}{2}\,A^a_\mu\left(\Box\,g^{\mu\nu} - \partial^\mu\partial^\nu\right)A^a_\nu \\ &- \frac{g}{2}\,f_{abc}\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right)A^b_\mu\,A^c_\nu - \frac{g^2}{4}\,f_{abe}f_{cde}\,A^\mu_a\,A^\nu_b\,A^c_\mu\,A^d_\nu\,, \end{aligned} \tag{2.1.18}$$

where $\cong$ means 'up to surface terms in the action', e.g. $\partial_\mu A^a_\nu\,\partial^\mu A^\nu_a \cong -A^a_\nu\,\Box\,A^\nu_a$ after partial integration. In contrast to the Abelian theory, where the F^2 term only produces a photon propagator, we can see that in the non-Abelian case we end up with the gluon propagator, a three-gluon interaction $\sim A^3$ and a four-gluon interaction $\sim A^4$.

Feynman rules. The terms $\bar\psi\,(i\not{D} - m)\,\psi$ and $-\frac{1}{4}F^a_{\mu\nu}\,F^{\mu\nu}_a$ in the Lagrangian allow us to read off the **Feynman rules** for the tree-level correlation functions of the QFT. In particular, the action contains the 1PI (one-particle irreducible, see Sec. 2.2.2) quantities, which means the vertices and *inverse* propagators that define the theory (Fig. 2.1). The procedure is as follows: symmetrize the respective term in the action (if necessary), transform it to momentum space, split off the integrals, fields and symmetry factors, and multiply with i to get the Feynman rule for the propagator or vertex.

For example, the inverse **quark propagator** corresponds to the term $\bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi$. The Fourier transform of the fields is

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\psi(p)\,. \tag{2.1.19}$$

Abbreviating $\int_p = \int d^4p/(2\pi)^4$, the term in the action becomes

$$\int d^4x\,\bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi = \int\limits_{p'}\int\limits_{p} \bar{\psi}(p')\,(p\!\!\!/ - m)\,\psi(p) \int d^4x\, e^{i(p'-p)\cdot x} = \int\limits_{p} \bar{\psi}(p)\,(p\!\!\!/ - m)\,\psi(p)$$

and dividing by i, the inverse tree-level propagator is

$$S_0^{-1}(p) = -i\,(p\!\!\!/ - m) \qquad \Leftrightarrow \qquad S_0(p) = \frac{i\,(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}\,. \tag{2.1.20}$$

Likewise, the inverse **gluon propagator** can be read off from Eq. (2.1.18). Replacing $\Box \to -p^2$ and $\partial^\mu\partial^\nu \to -p^\mu p^\nu$, we find

$$(D_0^{-1})^{\mu\nu}(p) = ip^2 \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right). \tag{2.1.21}$$

The symmetry factor 1/2 does not enter in the Feynman rule. Here we encounter, however, a difficulty: the inverse gluon propagator is proportional to a transverse projector, which is not invertible and thus the gluon propagator does not exist. We will cure the problem in Sec. 2.2.3 by the Faddeev-Popov method, where we follow analogous steps as in QED and add gauge-fixing terms to the action (which will also introduce ghost fields). Before we get there, keep in mind that the gluon propagator is not yet well-defined.

The **quark-gluon vertex** comes from the term $g\,\bar{\psi} A\!\!\!/\,\psi$ induced by the covariant derivative. If we denote the incoming and outgoing quark momenta by p and p' and the incoming gluon momentum by q, we have

$$\int d^4x\,\bar{\psi}\,gA\!\!\!/\,\psi = \sum_a \int\limits_{p'}\int\limits_{p}\int\limits_{q} (2\pi)^4\,\delta^4(p'-p-q)\,\bar{\psi}(p')\,A^a_\mu(q)\,g\gamma^\mu\,\mathsf{t}_a\psi(p)\,,$$

so the tree-level vertex is $ig\gamma^\mu\,\mathsf{t}_a$.

The **three-gluon vertex** must be fully symmetric under exchange of any two legs, but this symmetry is not yet manifest in the A^3 term of Eq. (2.1.18). To this end, we abbreviate $\partial^{\mu\nu\rho} = \partial^\mu g^{\nu\rho} - \partial^\nu g^{\mu\rho}$ and write

$$f_{abc}\,(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)\,A^b_\mu\,A^c_\nu = f_{abc}\,(\partial^{\mu\nu\rho}A^a_\rho)\,A^b_\mu\,A^c_\nu = f_{abc}\,A^a_\mu\,A^b_\nu\,(\partial^{\mu\nu\rho}A^c_\rho) = \dots$$

In the last step we renamed the color indices and used $f_{abc} = f_{bca} = f_{cab}$. For three Lorentz indices there are $3! = 6$ possible permutations; $\partial^{\mu\nu\rho}$ is already antisymmetric in $\mu \leftrightarrow \nu$ so we only need to add the two remaining cyclic permutations:

$$\begin{aligned} \dots &= \frac{1}{3}\,f_{abc}\left[A^a_\mu\,A^b_\nu\,(\partial^{\mu\nu\rho}A^c_\rho) + A^a_\nu\,A^b_\rho\,(\partial^{\nu\rho\mu}A^c_\mu) + A^a_\rho\,A^b_\mu\,(\partial^{\rho\mu\nu}A^c_\nu)\right] \\ &= \frac{1}{3}\,f_{abc}\left[A^a_\mu\,A^b_\nu\,(\partial^{\mu\nu\rho}A^c_\rho) + (\partial^{\nu\rho\mu}A^a_\mu)\,A^b_\nu\,A^c_\rho + A^a_\mu\,(\partial^{\rho\mu\nu}A^b_\nu)\,A^c_\rho\right]. \end{aligned} \tag{2.1.22}$$

In the first line we renamed the Lorentz indices and in the second line the color indices. Now we can pull out $A^a_\mu(p_1)\,A^b_\nu(p_2)\,A^c_\rho(p_3)$ in momentum space and the term in the action becomes

$$\begin{aligned}&-\frac{ig}{6}\,f_{abc}\int\limits_{p_1}\int\limits_{p_2}\int\limits_{p_3}(2\pi)^4\,\delta^4(p_1+p_2+p_3)\,A^a_\mu(p_1)\,A^b_\nu(p_2)\,A^c_\rho(p_3)\\&\quad\times\Big[(p_1-p_2)^\rho g^{\mu\nu}+(p_2-p_3)^\mu g^{\nu\rho}+(p_3-p_1)^\nu g^{\rho\mu}\Big],\end{aligned}\tag{2.1.23}$$

from where we read off the Feynman rule for the vertex:

$$\Gamma^{\mu\nu\rho}_{3g,0}=g f_{abc}\Big[(p_1-p_2)^\rho g^{\mu\nu}+(p_2-p_3)^\mu g^{\nu\rho}+(p_3-p_1)^\nu g^{\rho\mu}\Big]\,.\tag{2.1.24}$$

The symmetry factor 1/6 does again not enter, and $p_1+p_2+p_3=0$. The resulting vertex is Bose-symmetric, i.e., symmetric under a combined exchange of any two momenta with corresponding Lorentz and color indices.

The same strategy applies to the **four-gluon vertex** from the A^4 term in (2.1.18), which is also not yet manifestly symmetric:

$$\begin{aligned}f_{abe}f_{cde}\,A^a_\mu\,A^b_\nu\,A^\mu_c\,A^\nu_d&=f_{abe}f_{cde}\,g^{\mu\rho}g^{\nu\sigma}A^a_\mu\,A^b_\nu\,A^c_\rho\,A^d_\sigma\\&=\frac{1}{2}\,f_{abe}f_{cde}\,(g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma})\,A^a_\mu\,A^b_\nu\,A^c_\rho\,A^d_\sigma\,.\end{aligned}\tag{2.1.25}$$

Denoting $\Gamma^{\mu\nu\rho\sigma}=g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma}$, then with four Lorentz indices there are $4!=24$ possible permutations of $(\mu\nu\rho\sigma)\equiv(1234)$:

$$\begin{array}{cccccc}1234 & 3412 & 2314 & 1423 & 3124 & 2431\\1243 & 3421 & 2341 & 1432 & 3142 & 2413\\2134 & 4312 & 3214 & 4123 & 1324 & 4231\\2143 & 4321 & 3241 & 4132 & 1342 & 4213\end{array}\,.\tag{2.1.26}$$

The permutations in the first two columns are already covered because $\Gamma^{\mu\nu\rho\sigma}=-\Gamma^{\mu\nu\sigma\rho}$, etc., so we only need to add (2314) and (3124):

$$\begin{aligned}f_{abe}f_{cde}\,A^a_\mu\,A^b_\nu\,A^\mu_c\,A^\nu_d&=\frac{1}{6}\,f_{abe}f_{cde}\Big[(g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma})\,A^a_\mu\,A^b_\nu\,A^c_\rho\,A^d_\sigma\\&\qquad+(g^{\nu\mu}g^{\rho\sigma}-g^{\rho\mu}g^{\nu\sigma})\,A^a_\nu\,A^b_\rho\,A^c_\mu\,A^d_\sigma\\&\qquad+(g^{\rho\nu}g^{\mu\sigma}-g^{\mu\nu}g^{\rho\sigma})\,A^a_\rho\,A^b_\mu\,A^c_\nu\,A^d_\sigma\Big]\\&=\frac{1}{6}\,A^a_\mu\,A^b_\nu\,A^c_\rho\,A^d_\sigma\Big[f_{abe}f_{cde}\,(g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma})\\&\qquad+f_{bce}f_{ade}\,(g^{\nu\mu}g^{\rho\sigma}-g^{\rho\mu}g^{\nu\sigma})\\&\qquad+f_{cae}f_{bde}\,(g^{\rho\nu}g^{\mu\sigma}-g^{\mu\nu}g^{\rho\sigma})\Big].\end{aligned}\tag{2.1.27}$$

Together with $-g^2/4$ from (2.1.18), the combined symmetry factor for the A^4 term is indeed 1/24. The resulting four-gluon vertex is Bose-symmetric and given by

$$\begin{aligned}\Gamma^{\mu\nu\rho\sigma}_{4g,0}=-ig^2\Big[&f_{abe}f_{cde}\,(g^{\mu\rho}g^{\nu\sigma}-g^{\nu\rho}g^{\mu\sigma})\\&+f_{ace}f_{bde}\,(g^{\mu\nu}g^{\rho\sigma}-g^{\nu\rho}g^{\mu\sigma})\\&+f_{ade}f_{cbe}\,(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma})\Big]\,.\end{aligned}\tag{2.1.28}$$

QCD action. Putting everything together, the resulting QCD action constructed from the fields ψ, $\overline{\psi}$ and A_a^μ has the most general form that is invariant under Poincaré transformations, invariant under local gauge transformations, and renormalizable:

$$S_{QCD} = \int d^4x\, \mathcal{L}_{QCD}\,, \qquad \mathcal{L}_{QCD} = \overline{\psi}(x)\,(i\not{D} - \mathsf{M})\,\psi(x) - \tfrac{1}{4} F^a_{\mu\nu}\, F_a^{\mu\nu}\,. \tag{2.1.29}$$

The summation over the Dirac, color and flavor indices of the quarks is again implicit, and we generalized the quark mass m to a quark mass matrix $\mathsf{M} = \text{diag}\,(m_1 \dots m_{N_f})$. Some further remarks:

▪ Eq. (2.1.29) also conserves **charge conjugation** and **parity**, where the charge conjugation operation is defined by

$$\psi'_\alpha = \overline{\psi}_\beta\, C_{\beta\alpha}\,, \qquad \overline{\psi}'_\alpha = C_{\alpha\beta}\,\psi_\beta\,, \qquad C = i\gamma^2\,\gamma^0 \tag{2.1.30}$$

and the parity transformation by

$$\psi'(x') = \gamma^0\psi(x)\,, \qquad \overline{\psi}'(x') = \overline{\psi}(x)\,\gamma^0\,, \qquad x' = (t, -\boldsymbol{x})\,. \tag{2.1.31}$$

Since CPT is always conserved, this implies that the QCD action is also invariant under time reversal.

▪ In principle, another gauge-invariant and renormalizable (but parity-violating) term could appear in the Lagrangian, namely a topological charge density:

$$\mathcal{Q}(x) = \frac{g^2}{8\pi^2}\,\text{Tr}\left\{F_{\mu\nu}\,\widetilde{F}^{\mu\nu}\right\} \qquad \text{with} \quad \widetilde{F}^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,, \tag{2.1.32}$$

where $\widetilde{F}^{\mu\nu}$ is the dual field strength tensor. The resulting 'θ term' in the Lagrangian $\mathcal{L} + \theta\mathcal{Q}(x)$ violates parity and would give rise to an electric dipole moment of the neutron, whose experimental upper limit is however tiny ($\theta \leq 10^{-10}$). So it would seem that QCD *does* conserve parity; unfortunately, even if we started with $\theta = 0$ in QCD, the CP-violating weak interactions would renormalize it to $\theta \neq 0$. There are several possible scenarios how $\theta = 0$ could be enforced beyond the Standard Model, e.g. by promoting θ to a field (axions). Then again, CP *must* have been violated in the early universe, because otherwise the Big Bang would have created matter and antimatter in equal portions, which would have annihilated and resulted in a radiation universe without matter. This leads to the **strong CP problem**. On the other hand, since $\mathcal{Q} = \partial_\mu K^\mu$ can be written as the divergence of the Chern-Simons current K^μ, it only contributes a surface term to the action and in principle we could discard it (unless topological gauge field configurations play a role).

▪ We could have defined the gluon fields so that they absorb the coupling constant g (i.e., by replacing $A \to A/g$ and $F \to F/g$). From Eqs. (2.1.13), (2.1.18) and (2.1.29) we see that the only place in the Lagrangian where the coupling would then appear is in front of the gluon kinetic term, as a prefactor $1/g^2$. This shows that the sign of g is physically irrelevant.

Quark masses and flavor structure. With regard to the flavor structure, we can simply ignore the gluons since they are flavor independent. The quark-gluon interaction is flavor-blind, and the distinction between different quarks only comes from their masses. If the masses of all quark flavors were equal, the Lagrangian would have an additional $SU(N_f)$ flavor symmetry. This is not realized in nature, where

$$m_u \sim m_d \sim 2\dots 6\,\text{MeV}, \qquad m_s \sim 100\,\text{MeV}, \qquad \begin{array}{l} m_c \sim 1.3\ \text{GeV}, \\ m_b \sim 4.2\ \text{GeV}, \\ m_t \sim 173\ \text{GeV}. \end{array} \tag{2.1.33}$$

These **current-quark masses** have their origin in the Higgs sector and from the point of view of QCD they are external parameters that enter through the quark mass matrix $\mathsf{M} = \text{diag}(m_1 \dots m_{N_f})$. Because M is diagonal in flavor space, the flavor pieces in the Lagrangian simply add up: $\bar\psi\,\mathsf{M}\,\psi = \sum_f m_f\,\bar\psi_f\,\psi_f$. The flavor structure of the Lagrangian is crucial for the properties of hadrons and we will return to it in Chapter 3.

Infinitesimal gauge transformations. For later convenience it is useful to work out the infinitesimal transformations of the fields. The covariant derivative as defined in Eq. (2.1.3) acts on fields that transform under the fundamental representations of $SU(3)_C$, i.e., the group elements. When acting on elements of the algebra (those containing the matrix generators t_a, for example ε, A_μ or $F_{\mu\nu}$), we need an additional commutator in its definition: $D_\mu = \partial_\mu - ig\,[A_\mu, \cdot\,]$, or written in components:

$$\begin{aligned} (D_\mu \varepsilon)^a &= (\partial_\mu \varepsilon - ig\,[A_\mu, \varepsilon])^a = \partial_\mu \varepsilon^a - ig\,A_\mu^c\,\varepsilon^b\,i f_{cba} \\ &= (\partial_\mu\,\delta_{ab} - g f_{abc}\,A_\mu^c)\,\varepsilon^b = D_\mu^{ab}\,\varepsilon^b\,. \end{aligned} \tag{2.1.34}$$

In the fundamental representation, the group generators are the Gell-Mann matrices; in the adjoint representation they are given by $(\mathsf{t}_c)_{ab} = -i f_{abc}$. Inserting this into Eq. (2.1.3), we see that D_μ^{ab} is the covariant derivative in the adjoint representation:

$$(D_\mu)_{ab} = (\partial_\mu - igA_\mu)_{ab} = \partial_\mu\,\delta_{ab} - igA_\mu^c\,(\mathsf{t}_c)_{ab} = \partial_\mu\,\delta_{ab} - g f_{abc} A_\mu^c\,. \tag{2.1.35}$$

In an Abelian gauge theory such as QED, the commutator vanishes and $D_\mu^{ab} = \partial_\mu\,\delta_{ab}$ is the ordinary partial derivative.

With $U = 1 + i\varepsilon$, the infinitesimal gauge transformation of the fields is given by

$$\begin{aligned} \psi' &= U\psi \approx (1 + i\varepsilon)\,\psi\,, \\ \bar\psi' &= \bar\psi\,U^\dagger \approx \bar\psi\,(1 - i\varepsilon)\,, \\ A'_\mu &= U A_\mu U^\dagger + \frac{i}{g}\,U(\partial_\mu U^\dagger) \approx A_\mu + i\,[\varepsilon, A_\mu] + \frac{1}{g}\,\partial_\mu\varepsilon = A_\mu + \frac{1}{g} D_\mu\varepsilon\,, \\ F'_{\mu\nu} &= U\,F_{\mu\nu}\,U^\dagger \approx F_{\mu\nu} + i\,[\varepsilon, F_{\mu\nu}]\,, \end{aligned} \tag{2.1.36}$$

from where we obtain:

$$\delta\psi = i\varepsilon\,\psi\,, \qquad \delta\bar\psi = -i\bar\psi\,\varepsilon\,, \qquad \delta A_\mu = \frac{1}{g} D_\mu\varepsilon\,, \qquad \delta F_{\mu\nu} = i\,[\varepsilon, F_{\mu\nu}]\,. \tag{2.1.37}$$

Classical equations of motion. The classical Euler-Lagrange equations of motion follow from the action principle:

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi) \quad \Rightarrow \quad \delta S[\phi] = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi}\, \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \delta(\partial_\mu \phi) \right)$$
$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi(x) = 0\,,$$

which means that the functional derivative of the action vanishes:

$$\frac{\delta S[\phi]}{\delta \phi(x)} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0\,. \tag{2.1.38}$$

If the action contains several fields, there is one equation of motion for each component: $\delta S[\phi_1, \dots \phi_n]/\delta\phi_i(x) = 0$.

As a reminder, the **functional derivative** $\delta F[\phi]/\delta\phi(x)$ of a functional $F[\phi]$ is defined as

$$F[\phi + \delta\phi] = F[\phi] + \delta F[\phi] = F[\phi] + \int_{-\infty}^{\infty} dx\, \frac{\delta F[\phi]}{\delta\phi(x)}\, \delta\phi(x)\,, \tag{2.1.39}$$

where the last term is the continuum version of $\sum_i (\delta F/\delta\phi_i)\, \delta\phi_i$ written for one spacetime dimension. Here are some examples:

$F[\phi]$	$F[\phi+\delta\phi]$	$\frac{\delta F[\phi]}{\delta\phi(x)}$
$\int dx\, \phi(x)\, J(x)$	$\int dx\, (\phi + \delta\phi)\, J = F[\phi] + \int dx\, J\, \delta\phi$	$J(x)$
$\int dx\, f(\phi(x))\, J(x)$	$\int dx\, [f(\phi) + f'(\phi)\, \delta\phi]\, J$	$f'(\phi(x))\, J(x)$
$\int dx\, f(\phi(x), \phi'(x))$	$\int dx \left[f(\phi, \phi') + \frac{\partial f}{\partial \phi}\, \delta\phi + \frac{\partial f}{\partial \phi'}\, \delta\phi' \right]$ $= F[\phi] + \int dx \left[\frac{\partial f}{\partial \phi} - \frac{d}{dx} \frac{\partial f}{\partial \phi'} \right] \delta\phi$	$\frac{\partial f}{\partial \phi(x)} - \frac{d}{dx} \frac{\partial f}{\partial \phi'(x)}$
$\int_0^\infty dx\, \phi(x) = \int_{-\infty}^{\infty} dx\, \phi(x)\, \Theta(x)$	$F[\phi] + \int dx\, \Theta(x)\, \delta\phi(x)$	$\Theta(x)$
$\exp\left[i \int dx\, \phi(x)\, J(x)\right]$	$\exp\left[i \int dx\, (\phi(x) + \delta\phi(x))\, J(x)\right]$ $= F[\phi]\, \left(1 + i \int dx\, J(x)\, \delta\phi(x)\right)$	$iJ(x)\, \exp\left[i \int dy\, \phi(y)\, J(y)\right]$
$\phi(z) = \int dx\, \phi(x)\, \delta(x-z)$	$F[\phi] + \int dx\, \delta\phi(x)\, \delta(x-z)$	$\delta(x-z)$
$f(\phi(z))$	$f(\phi(z)) + f'(\phi(z))\, \delta\phi(z)$ $= F[\phi] + \int dx\, f'(\phi(x))\, \delta\phi(x)\, \delta(x-z)$	$f'(\phi(x))\, \delta(x-z)$
$\phi'(z) = \int dx\, \phi'(x)\, \delta(x-z)$ $= -\int dx\, \phi(x)\, \delta'(x-z)$	$F[\phi] - \int dx\, \delta\phi(x)\, \delta'(x-z)$	$-\delta'(x-z)$

Let us work out the classical equations of motion of QCD defined by the Lagrangian $\mathcal{L} = \overline{\psi}\left(i\overrightarrow{\partial\!\!\!/} + g A\!\!\!/ - \mathsf{M}\right)\psi - \frac{1}{4}\,F^a_{\mu\nu}\,F_a^{\mu\nu}$. Although they are not directly relevant for our purposes, they will later enter in the quantum equations of motion (Sec. 2.2.2) and conservation laws (Sec. 3.1). If we take the derivatives of $\mathcal{L}$ with respect to ψ and $\overline{\psi}$,

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} = (iD\!\!\!\!/ - \mathsf{M})\,\psi\,, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0\,, \quad \frac{\partial\mathcal{L}}{\partial\psi} = \overline{\psi}\,(gA\!\!\!/ - \mathsf{M})\,, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \overline{\psi}\,i\gamma^\mu \tag{2.1.40}$$

we obtain the **Dirac equations** for the quark and antiquark fields:

$$\frac{\delta S}{\delta\overline{\psi}} = (iD\!\!\!\!/ - \mathsf{M})\,\psi = 0\,, \qquad \frac{\delta S}{\delta\psi} = \overline{\psi}\,(-i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ - \mathsf{M}) = 0\,. \tag{2.1.41}$$

For the gluons, we first work out the derivatives of the field-strength tensor:

$$\frac{\partial F^a_{\mu\nu}}{\partial A^c_\rho} = g f_{abc}\left(A^b_\mu\,\delta^\rho_\nu - A^b_\nu\,\delta^\rho_\mu\right), \qquad \frac{\partial F^a_{\mu\nu}}{\partial(\partial_\rho A^c_\sigma)} = \left(\delta^\rho_\mu\,\delta^\sigma_\nu - \delta^\rho_\nu\,\delta^\sigma_\mu\right)\delta_{ac}\,. \tag{2.1.42}$$

With the product rule we then obtain

$$\frac{\partial\mathcal{L}}{\partial A^a_\mu} = g\,\overline{\psi}\,\gamma^\mu \mathsf{t}_a\psi + g f_{abc}\,A^c_\nu\,F_b^{\mu\nu}\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\nu A^a_\mu)} = F_a^{\mu\nu} \tag{2.1.43}$$

and finally

$$\frac{\delta S}{\delta A^a_\nu} = g f_{abc}\,A^c_\nu\,F_b^{\mu\nu} - \partial_\nu\,F_a^{\mu\nu} + g\,\overline{\psi}\,\gamma^\mu \mathsf{t}_a\psi = 0\,. \tag{2.1.44}$$

The first two terms on the r.h.s. can be combined to

$$-\left(\partial_\nu\,\delta_{ab} - g f_{abc}\,A^c_\nu\right)F_b^{\mu\nu} = -D^{ab}_\nu\,F_b^{\mu\nu} = -(D_\nu\,F^{\mu\nu})^a\,, \tag{2.1.45}$$

whereas the last term is the vector current corresponding to the global $SU(3)_C$ transformation: $J^\mu_a = \overline{\psi}\,\gamma^\mu \mathsf{t}_a\psi$. Then the quantity $J^\mu = \sum_a J^\mu_a\,\mathsf{t}_a$ lives in the Lie algebra and Eq. (2.1.44) becomes

$$D_\nu\,F^{\mu\nu} = g\,J^\mu\,. \tag{2.1.46}$$

These are the **classical Yang-Mills equations** for the gluon field, i.e., the Maxwell equations in the non-Abelian theory. They are the direct generalization from electrodynamics, where the covariant derivative in the adjoint representation would reduce to the ordinary derivative.

It is not too much of a stretch to ask whether there is also a generalization of the Maxwell equation for the dual field strength tensor $\widetilde{F}^{\mu\nu}$. Indeed we find

$$\begin{aligned} D_\nu\,\widetilde{F}^{\mu\nu} &= \tfrac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,D_\nu\,F_{\alpha\beta} \\ &= \tfrac{1}{6}\left(\varepsilon^{\mu\nu\alpha\beta} + \varepsilon^{\mu\alpha\beta\nu} + \varepsilon^{\mu\beta\nu\alpha}\right)D_\nu\,F_{\alpha\beta} \\ &= \tfrac{1}{6}\,\varepsilon^{\mu\nu\alpha\beta}\left(D_\nu\,F_{\alpha\beta} + D_\alpha\,F_{\beta\nu} + D_\beta\,F_{\nu\alpha}\right) = 0\,, \end{aligned} \tag{2.1.47}$$

where the parenthesis vanishes due to the **Bianchi identity**, which follows from the Jacobi identity for the generators, Eq. (A.1.3). Similarly, one can establish covariant current conservation $D_\mu J^\mu = 0$ for the solutions of the equations of motion.