Chapter 4

Currents and scattering

The goal of this remaining chapter is to investigate hadronic scattering processes, either with leptons or with other hadrons. These are important for illuminating the internal structure of hadrons: since there are no free quarks and gluons, we need to exploit the electromagnetic, weak and strong (in the sense of hadronic) interactions to resolve them experimentally. The basic structure observables of interest are form factors. When viewed from a distance, the proton looks like a point fermion that only carries a charge and a magnetic moment. When probed with short-wavelength photons (or other currents), it will reveal more and more of its composite nature which is encoded in the momentum dependence of its form factors.

4.1 Hadronic currents and form factors

Matrix elements. Form factors are the Lorentz-invariant dressing functions of the current matrix element $\langle \lambda' | j^\mu (0) | \lambda \rangle$ shown in Fig. 4.1. As in our earlier discussion in Section 2.2, the 'currents' of interest are vector, axialvector, scalar or pseudoscalar quark bilinears, and depending on their nature they will define the various form factors of hadrons. Let's deal with the kinematics first: the incoming hadron $\lambda$ carries a momentum $p_i$ and the outgoing hadron a momentum $p_f$. For elastic form factors the hadron remains on its mass shell: $p_i^2 = p_f^2 = M^2$. The three-point function is then described by two independent momenta; we will work with the average momentum $p = (p_i + p_f)/2$ and the momentum transfer $q = p_f - p_i$. If we invert these relations we obtain:

$$
\begin{align*}
p_f &= p + \frac{q}{2} \\
p_i &= p - \frac{q}{2}
\end{align*}
$$

$$
\begin{align*}
p_f^2 &= p^2 + \frac{q^2}{4} + p \cdot q = M^2 \\
p_i^2 &= p^2 + \frac{q^2}{4} - p \cdot q = M^2
\end{align*}
\Rightarrow
p \cdot q = 0
\Rightarrow
p^2 = M^2 - \frac{q^2}{4}.
\tag{4.1}
$$

Hence, in the elastic case both variables $p \cdot q$ and $p^2$ are fixed and the only remaining independent variable is the squared momentum transfer $q^2$. Since $q^2$ is negative in the spacelike region, one usually works with the spacelike momentum transfer $Q^2 = -q^2$, or equally $\tau := Q^2/(4M^2)$. 

A possible starting point of the discussion is the (non-local) correlation function of two quarks and two onshell hadrons,\footnote{1Which, in the case of baryons, is also the residue of the quark eight-point function after inserting two completeness relations for the incoming and outgoing baryonic states.} pictured in Fig. 4.1:

$$G_{\alpha\beta}(x_1, x_2, p_f, p_i) := \langle \lambda | T \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) | \lambda \rangle = \langle \lambda' | T \psi_\alpha(z) \bar{\psi}_\beta(-\frac{z}{2}) | \lambda \rangle e^{i q \cdot x}. \quad (4.2)$$

As in (2.74), we exploited the transformation behavior of the quark field operators and one-particle states under a translation, which allowed us to pull out the dependence on the total coordinate $x$ that only enters through a phase. The correlation function depends only on the relative coordinate $z = x_1 - x_2$, and the dependence on the momenta $p_i$ and $p_f$ only enters via the hadron states. The Dirac-flavor contraction

$$- \Gamma_{\beta\alpha} t_a G_{\alpha\beta}(x, x, p_f, p_i) = \langle \lambda' | j^T_a(x) | \lambda \rangle = \langle \lambda' | j^T_a(0) | \lambda \rangle e^{i q \cdot x} \quad (4.3)$$

yields then the gauge-invariant matrix element of the current that depends on the two momenta $p$ and $q$. In analogy to (2.77) we can write down its most general structure in momentum space, where the Lorentz-invariant coefficients define the form factors of the hadron under consideration.

The most prominent examples are the Pauli and Dirac form factors that constitute a spin-$1/2$ baryon’s vector current matrix element:

$$\langle p_f, \sigma' | V^\mu_a(0) | p_i, \sigma \rangle = \bar{u}_{\sigma'}(p_f) \left[ \gamma^\mu F_1(q^2) + \sigma^{\mu\nu} \frac{i q^\nu}{2 M} F_2(q^2) \right] u_{\sigma}(p_i) t^a_{ij}. \quad (4.4)$$

Here, $u(p_i)$ and $\bar{u}(p_f)$ are onshell Dirac spinors with the normalization $\bar{u}_{\sigma'}(p) u_{\sigma}(p) = 2M \delta_{\sigma\sigma'}$. $\sigma$ and $\sigma'$ are the spin indices of the incoming and outgoing baryon, and $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. The dimensionless form factors depend only on $q^2$ because the baryon is onshell and the remaining Lorentz invariants are fixed. If we restrict ourselves to

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.1.png}
\caption{Current matrix element of a hadron, viewed as the Dirac-flavor contraction of the four-point function $G$ from Eq. (4.2). At a timelike meson pole location, the matrix element factorizes into the meson’s decay constant, a propagator and the meson-hadron decay amplitude.}
\end{figure}
two flavors and work with an isoscalar current \( V^\mu = \bar{\psi} \gamma^\mu \psi = \bar{u} \gamma^\mu u + \bar{d} \gamma^\mu d \), we obtain the isoscalar form factors; the isovector form factors follow from inserting an isovector current \( V_3^\mu = \bar{\psi} \gamma^\mu t_3 \psi = \frac{1}{2} (\bar{u} \gamma^\mu u - \bar{d} \gamma^\mu d) \). The electromagnetic form factors are obtained from the quark charge matrix

\[
Q = \begin{pmatrix} q_u & 0 \\ 0 & q_d \end{pmatrix} = \frac{1}{6} + \frac{\tau_3}{2} \tag{4.5}
\]

and they are therefore linear combinations of isoscalar and isovector form factors.

Why are there exactly two form factors in Eq. (4.4)? Consider the most generic form of a vector-spinor three-point function \( \Omega^\mu(p,q) \). Poincaré covariance and parity invariance allows in principle for 12 tensor structures, for example

\[
\{ \gamma^\mu, p^\mu, q^\mu \} \times \{ 1, \bar{p}, \bar{q}, [\bar{p}, \bar{q}] \}, \tag{4.6}
\]

or suitable combinations of those. After sandwiching between the onshell nucleon spinors \( \bar{u}(p_f) \) and \( u(p_i) \), we can exploit the Dirac equation \((\slashed{p} - M) u(p) = 0\) to eliminate all slashed elements:

\[
\bar{u}(p_f) \gamma^\mu u(p_i) = \bar{u}(p_f) (\slashed{p}_f - \slashed{p}_i) u(p_i) = 0, \tag{4.7}
\]

\[
\bar{u}(p_f) \slashed{p} u(p_i) = \bar{u}(p_f) (\slashed{p}_f + \slashed{p}_i) u(p_i) = M \bar{u}(p_f) u(p_i), \tag{4.8}
\]

\[
\bar{u}(p_f) [\gamma^\mu, \bar{q}] u(p_i) = 4 \bar{u}(p_f) (p^\mu - M \gamma^\mu) u(p_i), \tag{4.9}
\]

and so on, so that we are left with \( \gamma^\mu, p^\mu \) and \( q^\mu \). Since \( q^\mu \) has the wrong \( C \)-parity, it must come with a factor \( p \cdot q \) which vanishes onshell.\(^2\) Hence, the vector form factors are the dressing functions of the remaining structures \( \gamma^\mu \) and \( p^\mu \). Alternatively, \( p^\mu \) can be expressed in terms of \( \gamma^\mu \) and \( \sigma^{\mu\nu} q_\nu = \frac{i}{2} [\gamma^\mu, \bar{q}] \) with the help of the Gordon identity (4.9) which leads to the form in (4.4).

The same considerations can also be used to derive the axialvector, scalar and pseudoscalar matrix elements of a baryon. Written in the form (4.4), the square bracket corresponding to the axialvector current \( A^\mu(0) \), the pseudoscalar density \( P(0) \) and the scalar density \( S(0) \) has to be replaced with

\[
\gamma^\mu \gamma_5 G_A(q^2) + \gamma_5 \frac{q^\mu}{2M} G_5(q^2), \quad G_5(q^2) i \gamma_5, \quad G_S(q^2), \tag{4.11}
\]

respectively. \( G_A \) is the axial form factor and \( G_P \) the induced pseudoscalar form factor of a spin-1/2 baryon, \( G_5 \) is the pseudoscalar form factor and \( G_S \) the scalar form factor. Going further, one could also derive the current matrix elements for spin-3/2 baryons (here the Dirac spinors would have to be replaced by Rarita-Schwinger spinors, which produces more tensor structures and hence more form factors), or transition matrix elements between baryons with different spins, meson form factors, etc.

\(^2\) Charge conjugation imposes the conditions

\[
C \Omega(−p,q)^T C^T = C \Omega(p,q), \quad C \Omega^\mu(−p,q)^T C^T = C \Omega^\mu(p,q) \tag{4.10}
\]

for any of the quantities \( \Omega(\mu)(p,q) \), where \( C = i\gamma^2 \gamma^\mu \) is the charge-conjugation matrix and \( C = ±1 \) the \( C \)-parity eigenvalue (vector: \( C = −1 \), scalar: \( C = +1 \) and so on, see below Eq. (2.95)). Tensor structures with the 'wrong' \( C \)-parity must be equipped with a factor \( p \cdot q \) to ensure the correct \( C \)-parity of the total three-point function.
Current conservation and Goldberger-Treiman relation. In analogy to the discussion of Eq. (2.78), we want to work out the implications of vector current conservation and PCAC for the current matrix elements. From Eq. (4.3) we have

$$\partial_\mu \langle \lambda' | V^\mu(x) | \lambda \rangle = \langle \lambda' | V^\mu(0) | \lambda \rangle \partial_\mu e^{iq \cdot x} = iq_\mu \langle \lambda' | V^\mu(0) | \lambda \rangle e^{iq \cdot x} \frac{1}{4} = 0,$$

(4.12)

which means that the vector current matrix element must be transverse with respect to the transfer momentum $q_\mu$. Eq. (4.4) already satisfies that constraint because $\sigma^{\mu\nu} q_\mu q_\nu = 0$ and $\bar{u}(p_f)\not\!q u(p_i) = 0$. In the axialvector case we obtain

$$\partial_\mu \langle \lambda' | A^\mu(x) | \lambda \rangle = iq_\mu \langle \lambda' | A^\mu(0) | \lambda \rangle e^{iq \cdot x} \frac{1}{4} = 2m \langle \lambda' | P(0) | \lambda \rangle e^{iq \cdot x},$$

(4.13)

which entails that the axial and pseudoscalar form factors are related:

$$G_A + \frac{q^2}{4M^2} G_P = \frac{m}{M} G_5.$$

(4.14)

It is apparent from Fig. 4.1 that the correlation function $G_{\alpha\beta}$ will exhibit timelike meson poles for $q^2 = m_\pi^2$ because it contains a $q\bar{q}$ (and also a $NN$) pair that is compatible with all meson quantum numbers. If we consider the pseudoscalar current contraction, the respective pseudoscalar residue is the one in Eq. (2.77), whereas the residue on the nucleon side contains the pion-nucleon coupling constant $g_{\pi NN}$. Therefore, $G_5$ has the form\footnote{The minus sign ensures that $G_5(q^2)$ is positive for spacelike momenta $q^2 < 0$, and the factor 2 corresponds to the choice $\tau_3 = 2t_\alpha$ for the flavor generator of the pion: $G_5 t_\alpha \sim 2G_{\pi NN} t_\alpha = G_{\pi NN} \tau_3$.}

$$G_5(q^2) = -\frac{r_\pi}{q^2 - m_\pi^2} 2G_{\pi NN}(q^2) = \frac{-f_\pi m_\pi^2}{2m} \frac{1}{q^2 - m_\pi^2} 2G_{\pi NN}(q^2).$$

(4.15)

Here we defined an effective pion-nucleon coupling $G_{\pi NN}(q^2)$ that absorbs all further pseudoscalar pole contributions from the spectral representation and reduces to $G_{\pi NN}(q^2 = m_\pi^2) = g_{\pi NN}$ at the pion pole. Combined with Eq. (4.14), one arrives at the Goldberger-Treiman relation that connects the axial charge $G_A(0) = g_A$ with the pion-nucleon coupling:

$$g_A = \frac{f_\pi}{M} G_{\pi NN}(0) \xrightarrow{\text{chiral limit}} \frac{f_\pi}{M} g_{\pi NN}.$$  

(4.16)

Meson resonances. Similarly, the vector form factors $F_1(q^2)$ and $F_2(q^2)$ have $(1^{--})$ vector-meson poles at $q^2 = m_\rho^2$ (or $m_\omega^2$, depending on whether one studies the isovector or isoscalar channel), and their residues will be the products of the $\rho/\omega$–nucleon couplings together with the $\rho/\omega$–meson decay constants. The axial form factor $G_A(q^2)$ contains only axialvector $(1^{++})$ poles, and the scalar form factor has scalar poles $(0^{++})$. In fact, since only the pion is stable with respect to the strong interaction, all other mesons have non-zero hadronic decay widths. Their poles will be shifted into the complex $q^2$ plane (second Riemann sheet) and only produce bumps on the timelike $q^2$ axis. In general, the spectral representation will also contain all intermediate $n$–particle states containing two pions ($p \to \pi\pi$), three pions ($\omega \to \pi\pi\pi$), $K\bar{K}$, etc.
The situation is illustrated in Fig. 4.2 for a generic nucleon electromagnetic (more precisely: vector-isovector) form factor with \(\rho\)-meson bumps; a similar picture with appropriate \(J^{PC}\) poles would arise for other types of form factors as well. The form factor’s momentum dependence in the spacelike domain \((Q^2 = -q^2 > 0)\) can be extracted from elastic electron-nucleon scattering as long as the one-photon exchange process is dominant (more on that below). The timelike region above \(p\bar{p}\) production threshold \((q^2 > 4M^2)\) can be accessed in \(e^+e^-\) annihilation. However, meson resonances should be most pronounced in the window \(q^2 \sim 0 \ldots 4\text{ GeV}^2\) which is experimentally not accessible; in the deep timelike region the resonance peaks are already washed out. Fortunately, precise data are available for the pion electromagnetic form factor which should display a similar resonance structure as in the nucleon case. Here the unphysical window is much smaller \((q^2 = 0 \ldots 4m^2_\pi \approx 0.08\text{ GeV}^2)\) and the resonance peaks are indeed directly visible in the data, with a similar shape as in Fig. 4.2.

**Dispersion relations.** The timelike resonance structure can be connected with the spacelike behavior of the form factors via dispersion relations. Causality implies that the form factors are analytic everywhere in the complex \(Q^2\) plane except for a branch-cut singularity starting at \(q^2 = 4m^2_\pi\) (due to intermediate two-pion and multiparticle states) and extending to infinity. Cauchy’s formula then tells us that the form factor in the analytic region can be obtained from knowledge of its value on a closed contour, which can be deformed to encompass only the branch cut (see Fig. 4.3). Since the form factor is analytic everywhere else, the difference above and below the branch cut is proportional to its imaginary part, i.e., the discontinuity along the branch cut:

\[
F(z_0) = \frac{1}{2\pi i} \oint dz \frac{F(z)}{z-z_0} = \frac{1}{\pi} \int_{4m^2_\pi}^{\infty} dz \frac{\text{Im} F(z)}{z-z_0}. \tag{4.17}
\]

Hence, knowledge of the spectral function \(\text{Im} F(z)\) along the cut would be sufficient to
determine the spacelike form factors as well. On the other hand, since the experimental knowledge is limited to $q^2 > 4M^2 \sim 4\text{ GeV}^2$, one usually has to make assumptions about the timelike analytic structure to extract such information.

**Cross section for elastic $eN$ scattering.** The nucleon’s electromagnetic form factors in the spacelike region $Q^2 \geq 0$ are experimentally extracted from elastic electron-nucleon scattering which we want to study in more detail. Now might be a good time to consult Appendix D: the scattering process is pictured in Fig. D.1, and we work with the variables defined in Eq. (D.4). Since the electron and nucleon are scattered elastically, the inelasticities $\omega$ and $k \cdot q$ vanish and $k^2 = M^2 \tau$ is fixed. Hence, the cross section depends only on the momentum transfer $\tau$ and the crossing variable $\nu$. The general form of the cross section for two- to $n$–particle scattering has the form

$$d\sigma = \frac{|M|^2 d\Phi}{4\sqrt{(p_i \cdot k_i)^2 - m_1^2 m_2^2}},$$

(4.18)

where $|M|^2$ is the invariant amplitude, $d\Phi$ is the phase space element and the denominator is the incoming flux factor.

Let’s start with the phase space. For two particles in the final state it is given by

$$d\Phi = \frac{d^3 p_f}{(2\pi)^3 2E_N'} \frac{d^3 k_f}{(2\pi)^3 2E'} \frac{1}{(2\pi)^4} \delta^4(p_i + k_i - k_f - p_f),$$

(4.19)

where $p_f$ and $k_f$ are the final nucleon and electron momenta, and $E_N'$ and $E'$ are their corresponding energies in the lab frame, cf. (D.9). Integration over $d^3 p_f$ removes the three-dimensional $\delta$–function for three-momentum conservation, and inserting $d^3 k_f = dE'E'^2 d\Omega$ yields

$$d\Phi = \frac{d\Omega}{(4\pi)^2} \frac{E'}{E_N'} dE' \delta(M + E - E' - E_N').$$

(4.20)

We can express the final nucleon energy by $E_N' = \sqrt{q^2 + W^2} = \sqrt{q^2 + M^2 + 4M^2 \omega}$ where, for elastic scattering, the energy-conservation constraint is satisfied for $\omega = 0$.  

**Figure 4.3:** Analytic structure of the form factor $F(Q^2)$ in the complex $Q^2$ plane and deformation of the integration contour.
Hence, we can rewrite the δ-function in the variable \( \omega \):

\[
\delta(M + E - E' - E_N) = \frac{E'_N}{2M^2} \delta(\omega) \quad \Rightarrow \quad d\Phi = \frac{d\Omega}{(4\pi)^2} \frac{E'}{2M^2} dE' \delta(\omega). \tag{4.21}
\]

On the other hand, we have

\[
\int dE' f(E') \frac{\delta(\omega)}{2M} = \frac{f(E')}{2M} \frac{d\omega}{\delta(\omega)} \left. \right|_{\omega=0} = \frac{f(E')}{1 + \frac{2E'}{M} \sin^2 \frac{\theta}{2}} \left. \right|_{\omega=0} = \frac{f(E') E'}{E}. \tag{4.22}
\]

Combining this with the flux factor \( 4p_i \cdot k_i = 4ME \), we arrive at

\[
\frac{d\sigma}{d\Omega} = \frac{|M|^2 E^2}{16\pi^2 E^2}. \tag{4.23}
\]

Next, we want to compute the invariant amplitude \(|M|^2\). For the scattering of unpolarized leptons from a point-like Dirac particle in Born approximation (one-photon exchange) it is given by the spin average

\[
|M|^2 = e^4 \frac{1}{q^4} \sum_{\text{spins}} |\bar{u}(k_f)\gamma^\mu u(k_i) \bar{u}(p_f)\gamma_\mu u(p_i)|^2 = e^4 \frac{1}{q^4} L^\mu\nu W_{\mu\nu} \tag{4.24}
\]

which factorizes in a leptonic and a hadronic part. The lepton tensor has the form

\[
L^\mu\nu = \frac{1}{2} \sum_{ss'} \bar{u}_s(k_f) \gamma^\mu u_s(k_i) \bar{u}_{s'}(k_i) \gamma^\nu u_{s'}(k_f) = \frac{1}{2} \text{Tr} \left[ \gamma^\mu \gamma^\nu \right] = 4 \left( k^\mu_T k^\nu_T + \frac{q^2}{4} T_{q\mu\nu}^\mu \right). \tag{4.25}
\]

The final step requires a little calculation; here we have used the transverse projector and the transversely projected average momentum:

\[
T_{q\mu\nu}^\mu := g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}, \quad k_T^\mu := T_{q\mu\nu} k_\nu = k^\mu - \frac{k \cdot q}{q^2} q^\mu. \tag{4.26}
\]

Since the electron scatters elastically, we have \( k \cdot q = 0 \) and therefore \( k_T^\mu = k^\mu \). The lepton tensor is transverse with respect to the photon momentum in both Lorentz indices which reflects the conservation of the leptonic vector current. For elastic scattering on the hadron side \((\omega = 0)\), the hadronic tensor for a structureless fermion has the analogous form

\[
W_{\mu\nu} = 4 \left( p_T^\mu p_T^\nu + \frac{q^2}{4} T_{q\mu\nu}^\mu \right), \tag{4.27}
\]

and with \( p_T \cdot k_T = p \cdot k \), \( k_T^2 = k^2 \), and \( T_{\mu\nu} T_{\mu\nu} = 3 \), their combination becomes

\[
L^\mu\nu W_{\mu\nu} = 16 \left[ (p \cdot k)^2 + \frac{q^2}{4} (k^2 + p^2) + 3 \frac{q^4}{16} \right] = 16M^4 (\nu^2 + \tau^2 - \tau). \tag{4.28}
\]

Setting \( e^2 = 4\pi\alpha \), the result for the invariant amplitude is

\[
|M|^2 = e^4 \frac{1}{q^4} L^\mu\nu W_{\mu\nu} = \frac{16\pi^2\alpha^2}{\tau^2} (\nu^2 + \tau^2 - \tau), \tag{4.29}
\]
and the differential cross section becomes
\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4M^2\tau^2} \frac{E'^2}{E^2} (\nu^2 + \tau^2 - \tau) = \frac{\alpha^2\cos^2\frac{\theta}{2}}{4E^2\sin^4\frac{\theta}{2}} \left( 1 + 2\tau \tan^2\frac{\theta}{2} \right) , \] (4.30)

where we exploited the relations (D.10–D.11) to arrive at the second form. The Mott cross section describes lepton scattering off a pointlike scalar particle in Born approximation; the bracket reflects the nucleon’s nature as a spin-\( \frac{1}{2} \) particle (which carries no internal structure at this point).

In order to take the composite nature of the nucleon into account, we have to replace the pointlike Dirac current with the general current matrix element
\[ \bar{u}(p_f)\gamma^\mu u(p_i) \rightarrow \bar{u}(p_f) \left( \gamma^\mu F_1(q^2) + \sigma^{\mu\nu} \frac{iq_{\nu}}{2M} F_2(q^2) \right) u(p_i) \] (4.31)
with Pauli and Dirac form factors \( F_1 \) and \( F_2 \). Here it is more convenient to work with the Sachs electric and magnetic form factors \( G_E := F_1 - \tau F_1 \), \( G_M := F_1 + F_2 \) since they will not produce interference terms \( \sim F_1 F_2 \) in the cross section. The invariant amplitude becomes
\[ |\mathcal{M}|^2 = \frac{16\alpha^2\pi^2}{\tau^2} \left[ \frac{G_E^2}{1 + \tau} (\nu^2 - \tau^2 - \tau) + 2\tau^2 G_M^2 \right] , \] (4.32)
and the resulting cross section is the Rosenbluth cross section:
\[ \frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left( \frac{G_E^2}{1 + \tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2} \right) . \] (4.33)

For a structureless fermion (\( F_1 = 1, F_2 = 0 \) or \( G_E = G_M = 1 \)) these formulas reduce to the previous forms (4.29) and (4.30). The Rosenbluth cross section allows us to extract the nucleon’s electromagnetic form factors under the assumption of one-photon exchange (which can actually cause problems for certain observables such as the ratio \( G_E/G_M \)). The strategy is then to measure the cross section at many angles \( \theta \) and many values of \( Q^2 \). If \( Q^2 \) becomes large, Eq. (4.33) has limited sensitivity to the electric form factor \( G_E \), which is therefore not so well known at large \( Q^2 \); for small \( Q^2 \) the cross section is less sensitive to the magnetic form factor \( G_M \) (except for backward angles). The form factors of the proton are directly accessible in \( ep \rightarrow ep \) scattering; those of the neutron are extracted from scattering on deuterium since there is no free neutron target in nature.

**Form factor phenomenology.** The Dirac and Pauli form factors at vanishing photon momentum encode the nucleon charges and their anomalous magnetic moments:
\[ F_1^p(0) = 1 , \quad F_1^n(0) = 0 , \quad F_2^p(0) = \kappa_p = 1.79 , \quad F_2^n(0) = \kappa_n = -1.91 . \]
Their slopes at \( Q^2 = 0 \) are related to Dirac and Pauli ‘charge radii’:
\[ F_1(Q^2) = F_1(0) - \frac{r_1^2}{6} Q^2 + \ldots , \quad F_2(Q^2) = F_2(0) \left[ 1 - \frac{r_2^2}{6} Q^2 + \ldots \right] . \] (4.34)
The Sachs form factors $G_E$ and $G_M$ contain the charges and magnetic moments

$$G_M^p(0) = \mu_p = 1 + \kappa_p = 2.79, \quad G_M^n(0) = \mu_n = \kappa_n = -1.91,$$

and one defines the electric and magnetic Sachs radii of proton and neutron accordingly.

Empirically, it turns out that the Sachs form factors can be reasonably well described by a dipole shape over a wide $Q^2$ range (except for $G_E^n$ which vanishes at the origin).

The 'dipole mass' $\Lambda$ can then be used to estimate the charge radii:

$$G_i(Q^2) \approx \frac{G_i(0)}{(1 + Q^2/\Lambda^2)^2}, \quad \Lambda = 0.84 \text{ GeV} \Rightarrow r_i \approx \frac{\hbar c}{\sqrt{12} \Lambda} \approx 0.8 \text{ fm} , \quad (4.35)$$

with $\hbar c = 0.197 \text{ GeV fm}$. Such a dipole behavior for the Sachs form factors agrees with perturbative QCD predictions but has been challenged by recent experiments: the data for $G_E^p/G_M^p$ fall off at larger $Q^2$ and signal a potential zero crossing.

The definition of the charge radii in (4.34) has its origin in the non-relativistic interpretation of a form factor as the Fourier transform of a charge distribution. Consider the scattering of an electron from a static, spinless source generated by the charge distribution $\rho(x)$ that generates the vector potential $A^\mu(x)$:

$$\Box A^\mu = j^\mu, \quad A^\mu = \begin{pmatrix} A^0 \\ 0 \end{pmatrix}, \quad j^\mu = \begin{pmatrix} e \rho \\ 0 \end{pmatrix}. \quad (4.36)$$

The invariant matrix element is given by

$$\mathcal{M} = ie \bar{u}(k_f) \gamma^\mu u(k_i) \int d^4xe^{-iq\cdot x} A_\mu(x) \cdot \left((2\pi) \delta(E - E') \frac{e}{q^2} F(q) \delta_\mu^0\right). \quad (4.37)$$

Since $A^\mu(x)$ is time-independent, its Fourier transform in time produces a $\delta-$function that enforces $E = E'$ for the lab energies of the incoming and outgoing electron. The Maxwell equation in (4.36) reduces to $\Delta A^0 = -e\rho$, and a partial integration yields:

$$\int d^3x e^{iq\cdot x} A^0(x) = \frac{e}{q^2} \int d^3x e^{iq\cdot x} \rho(x) =: \frac{e}{q^2} F(q). \quad (4.38)$$

The form factor, defined as the Fourier transformation of the charge density, therefore measures the deviation from the pointlike nature of the source. For a spherically symmetric charge distribution $\rho(x) = \rho(|x|) = \rho(r)$ that is normalized to $\int d^3x \rho(x) = 1$, the form factor at small $|q|$ can be expanded in

$$F(q) = \int d^3x \rho(x) \left(1 + i q \cdot x - \frac{(q \cdot x)^2}{2} + \ldots\right) = 1 - \frac{|q|^2}{6} \int dr \rho(r) r^4 + \ldots$$

The coefficient of the quadratic term is the mean-square radius of the 'charge cloud'.

In general, this picture is invalidated by the fact the proton is not static and will recoil; the recoil correction $E'/E \neq 1$ is already implemented in the relativistic Mott
cross section (4.30). In principle, one could interpret the form factors as the Fourier transforms of the charge and magnetization distributions in the *Breit frame* where the incoming and outgoing proton have opposite momenta \((p_f = -p_i = q/2)\) and hence the same energies, so that the photon transfers no energy and also \(E' = E\). This also implies \(Q^2 = |q|^2\), and one can show that a dipole falloff of the form factor corresponds to an exponential charge distribution:

\[
\rho(r) = \frac{\Lambda^3}{8\pi} e^{-\Lambda r} \iff F(q) = \int d^3x \, e^{iq \cdot x} \rho(x) = \frac{1}{(1 + |q|^2/\Lambda^2)^2}.
\] (4.39)

Furthermore, the vector current matrix element in the Breit frame reduces to the form

\[
\langle p_f, \sigma' | V^0 | p_i, \sigma \rangle = 2M G_E \delta_{\sigma'\sigma}, \quad \langle p_f, \sigma' | V | p_i, \sigma \rangle = G_M \chi_{\sigma'}^\dagger i\tau \times q \chi_\sigma,
\]

hence the name 'electric' and 'magnetic' form factors. However, since there is a different Breit frame for each value of \(Q^2\), the relation to charge densities in the lab frame (the rest frame of the nucleon) will suffer from relativistic boost corrections and hence the interpretation of the radii as actual charge and magnetization radii is not directly applicable. In general, while the Lorentz-invariant form factors uniquely specify the electromagnetic structure of a hadron, their physical interpretation in terms of spatial densities will usually depend on the reference frame.

**Magnetic moments in the quark model.** Current matrix elements encode the complicated nonperturbative substructure of hadrons and have only recently become amenable to first-principle calculations. Nevertheless, we can infer simple relations already from the nonrelativistic quark model. We have seen in Eq. (2.124) that the ground-state baryon octet states can be written as the combination of a flavor and a spin doublet:

\[
|\lambda\sigma\rangle = \mathcal{D}^\lambda \cdot \mathcal{D}^\sigma = \sum_{m=1}^2 \mathcal{D}_m^\lambda \mathcal{D}_m^\sigma, \quad \lambda \in \{p, n, \Sigma^+, \ldots\}, \quad \sigma \in \{\uparrow, \downarrow\}, \quad (4.40)
\]

combined with a fully symmetric spatial wave function and the antisymmetric color part. The flavor doublets are those in Table 2.3: for example, the one for the proton (\(\mathcal{D}^p\) in our present notation) is given in (2.117), and that of the neutron follows from exchanging \(u \leftrightarrow d\). The \(SU(2)\) spin doublets have the analogous form if we replace \(u\) by \(\uparrow\) and \(d\) by \(\downarrow\) for the baryon with spin up, and vice versa for the baryon with spin down. The index \(m\) in Eq. (4.40) sums over the mixed-symmetric and -antisymmetric entries. In the following we are only interested in the spin-flavor part. Its unit normalization is ensured via

\[
\langle \lambda'\sigma'|\lambda\sigma\rangle = \frac{1}{\mathcal{N}} \sum_{m'\lambda'\sigma'} (\mathcal{D}_{m'}^{\lambda'})^\dagger (\mathcal{D}_m^\lambda) (\mathcal{D}_{m'}^{\sigma'})^\dagger (\mathcal{D}_m^\sigma) = \delta_{\lambda'\lambda} \delta_{\sigma'\sigma},
\] (4.41)

from where the factor \(\mathcal{N}\) has to be determined. From Eq. (2.117) one can immediately verify that

\[
(\mathcal{D}_{m'}^p)^\dagger (\mathcal{D}_m^p) = (\mathcal{D}_{m'}^n)^\dagger (\mathcal{D}_m^n) = (\mathcal{D}_{m'}^{\uparrow})^\dagger (\mathcal{D}_m^{\uparrow}) = 2 \delta_{m'm},\quad (4.42)
\]
for example (with \(u^\dagger u = d^\dagger d = 1, \ u^\dagger d = d^\dagger u = 0\))

\[
(D_p^p)\dagger (D_1^p) = (u^\dagger d^\dagger u^\dagger - d^\dagger u^\dagger u^\dagger) (udu - dud) = 2, \quad \text{etc.} \quad (4.43)
\]

Inserting this in (4.41) yields

\[
\langle p^\uparrow | p^\uparrow \rangle = \langle n^\uparrow | n^\uparrow \rangle = \frac{4}{N} \text{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{8}{N} \Rightarrow N = 8. \quad (4.44)
\]

The expectation value of a generic flavor (\(F\)) and spin (\(\Gamma\)) operator is then

\[
\langle \lambda' \sigma' | F \Gamma | \lambda \sigma \rangle = \frac{3}{N} \sum_{m'm} (D^{\lambda'}_{m'})^\dagger F (D^\lambda_{m}) (D^{\sigma'}_{m'})^\dagger \Gamma (D^\sigma_{m}) = \frac{12}{N} \text{Tr} \left\{ F^{\lambda \lambda'} T^{\Gamma \sigma \sigma'} \right\}, \quad (4.45)
\]

which is understood in the sense that \(F\) and \(\Gamma\) act on the flavor and spin indices of the third quark in each doublet \(D\), and the factor 3 counts the three possible permutations. The trace in the last equation goes over the doublet indices. It is useful to work out the flavor and spin matrix elements of the \(SU(2)\) unit matrix and the Pauli matrix \(\tau_3\) for proton and neutron (use \(\tau_3 u = u, \ \tau_3 d = -d\)):

\[
1^\uparrow \uparrow = 1^{pp} = 1^{nn} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \tau_3^{\uparrow \uparrow} = \tau^{pp}_3 = \tau^{nn}_3 = -\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \quad (4.46)
\]

The matrix elements of the unit matrix are just those in (4.42). Their combination yields the two-flavor quark charge matrix, cf. Eq. (4.5):

\[
Q = \begin{pmatrix} q_u & 0 \\ 0 & q_d \end{pmatrix} = \frac{1}{6} + \frac{\tau_3}{2} \quad \Rightarrow \quad Q^{pp} = \frac{2}{3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q^{nn} = \frac{1}{3} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.47)
\]

from where one obtains the charges of proton and neutron:

\[
\langle p^\uparrow | Q | p^\uparrow \rangle = \frac{3}{2} \text{Tr} Q^{pp} = 1, \quad \langle n^\uparrow | Q | n^\uparrow \rangle = \frac{3}{2} \text{Tr} Q^{nn} = 0 \quad (4.48)
\]

as well as their magnetic moments:

\[
\langle p^\uparrow | Q \tau_3 | p^\uparrow \rangle = \frac{3}{2} \text{Tr} \left\{ Q^{pp} \tau^{\uparrow \uparrow}_3 \right\} = 1, \quad \langle n^\uparrow | Q \tau_3 | n^\uparrow \rangle = \frac{3}{2} \text{Tr} \left\{ Q^{nn} \tau^{\uparrow \uparrow}_3 \right\} = -\frac{2}{3}, \quad (4.49)
\]

apart from the remaining spatial integral. However, since the spatial part is taken to be identical for proton and neutron, the last relation yields the quark-model relation \(\mu_n/\mu_p = -\frac{2}{3}\) which is quite close to the experimental value \(-0.685\). Similarly, one can also work out the magnetic moments for the other ground-state octet members:

\[
\mu_{\Sigma^+} = 1, \quad \mu_{\Sigma^0} = \frac{1}{3}, \quad \mu_{\Sigma^-} = \mu_{\Xi^-} = \mu_{\Lambda} = -\frac{1}{3}, \quad \mu_{\Xi^0} = -\frac{2}{3}, \quad (4.50)
\]

and in principle also those of the decuplet baryons.