Appendix A

SU(N)

The group $SU(N)$ is probably the most important symmetry group in particle physics. $SU(2)$ encodes spin and isospin, $SU(3)$ describes both color and the physics of three light quark flavors, $SU(2) \times SU(2)$ is the universal cover of the (Euclidean) Lorentz group, etc. In the following we will collect some basic facts and useful formulas.

A.1 Basic properties of $SU(N)$

The group $SU(N)$ is the special unitary Lie group ($U^\dagger = U^{-1}$, $\det U = 1$) with $N^2 - 1$ real group parameters. The group element in a given representation can be written as

$$U = \exp\left(i \sum_{a=1}^{N^2-1} \varepsilon_a \mathsf{t}_a\right) = \exp\left(i\varepsilon\right), \tag{A.1}$$

where the $N^2 - 1$ generators t_a are hermitian and traceless. They form the basis of a Lie algebra with commutator relations

$$[\mathsf{t}_a, \mathsf{t}_b] = i f_{abc}\, \mathsf{t}_c\,, \tag{A.2}$$

where f_{abc} are the totally antisymmetric and real structure constants of $SU(N)$. For $SU(2)$ one has $f_{abc} = \epsilon_{abc}$; the structure constants of $SU(3)$ are given in Table A.1. The Jacobi identity for the generators,

$$[\mathsf{t}_a, [\mathsf{t}_b, \mathsf{t}_c]] + [\mathsf{t}_b, [\mathsf{t}_c, \mathsf{t}_a]] + [\mathsf{t}_c, [\mathsf{t}_a, \mathsf{t}_b]] = 0\,, \tag{A.3}$$

implies for the structure constants the relation $f_{abe}\, f_{cde} + f_{bce}\, f_{ade} + f_{cae}\, f_{bde} = 0$.

In the fundamental representation, the generators are $N \times N$ matrices. In the case of $SU(2)$, they are proportional to the Pauli matrices: $\mathsf{t}_a = \tau_a/2$, with

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

f_{abc}	abc
1	123
$\frac{1}{2}$	147, 246, 257, 345
$-\frac{1}{2}$	156, 367
$\frac{\sqrt{3}}{2}$	458, 678

d_{abc}	abc
$\frac{1}{\sqrt{3}}$	118, 228, 338
$\frac{1}{2}$	146, 157, 256, 344, 355
$-\frac{1}{2}$	247, 366, 377
$-\frac{1}{2\sqrt{3}}$	448, 558, 668, 778
$-\frac{1}{\sqrt{3}}$	888

Table A.1: Antisymmetric structure constants f_{abc} and symmetric symbols d_{abc} for the group $SU(3)$. The values for the remaining indices are obtained via permutation.

For $SU(3)$ they are given by the Gell-Mann matrices, $\mathsf{t}_a = \lambda_a/2$, with

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\ \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{A.5}$$

In the adjoint representation of $SU(N)$, the generators are given by $(\mathsf{t}_a)_{bc} = -i f_{abc}$, so they are $(N^2-1)\times(N^2-1)$ matrices. The generators satisfy

$$\mathrm{Tr}\,(\mathsf{t}_a\,\mathsf{t}_b) = T(R)\,\delta_{ab} \qquad \text{and} \qquad \Big(\sum_a \mathsf{t}_a^2\Big)_{ij} = C(R)\,\delta_{ij}\,, \tag{A.6}$$

where $T(R)$ is the Dynkin index and $C(R)$ the Casimir in the representation R:

- fundamental representation: $T(R) = \frac{1}{2}$, $C(R) = (N^2-1)/(2N)\,$,
- adjoint representation: $T(R) = C(R) = N$.

From Eq. (A.6) it follows that $T(R)D(A) = C(R)D(R)$, where $D(R)$ is the dimension of the representation R and $D(A)$ is the dimension of the adjoint (which defines the dimension of the group).

In the fundamental representation, one has the anticommutation relation

$$\{\mathsf{t}_a, \mathsf{t}_b\} = \frac{1}{N}\,\delta_{ab} + d_{abc}\,\mathsf{t}_c\,, \tag{A.7}$$

where the totally symmetric d_{abc} are collected in Table A.1 for the case of $SU(3)$; for $SU(2)$, they are zero. The structure constants are then obtained by taking traces of the generators in the fundamental representation, as follows from (A.2), (A.6) and (A.7):

$$f_{abc} = -2i\,\mathrm{Tr}\,([\mathsf{t}_a, \mathsf{t}_b]\,\mathsf{t}_c)\,, \qquad d_{abc} = 2\,\mathrm{Tr}\,(\{\mathsf{t}_a, \mathsf{t}_b\}\,\mathsf{t}_c)\,. \tag{A.8}$$

A.2 SU(N) representations

The group $SU(N)$ has N^2-1 generators t_a which form the basis of a Lie algebra defined by the commutator relations in (A.2). The group has rank $N-1$, so there are $N-1$ Casimir operators which commute with all generators and label the irreducible representations of $SU(N)$. A given irreducible representation is then defined by $N-1$ numbers. It has dimension D and defines a D-dimensional invariant subspace that can be visualized by collecting its basis states in a multiplet. Rank $N-1$ also entails that there are at most $N-1$ generators that commute with each other; they form the Cartan subalgebra (the maximal Abelian subalgebra) and label the states within the multiplet. The multiplets are therefore geometric structures in $N-1$ dimensions. The remaining generators are ladder operators that connect the states with each other. Here are examples for $N=2$, 3 and 4:

SU(2): The group $SU(2)$ describes angular momentum (spin, isospin, etc) and has three generators t_a (often called J_a). It has rank one, so there is one Casimir operator ($\mathsf{t}_a\,\mathsf{t}_a = \boldsymbol{J}^2$) whose eigenvalues $j(j+1)$ label the irreducible representations; the spin j can take integer and half-integer values. The Cartan subalgebra consists of only one generator (t_3) whose eigenvalues cover the interval $-j\dots j$ and label the states within each multiplet. Anticipating the notation for $SU(3)$, we denote the irreducible representations of $SU(2)$ by D^p, where $p=2j=0,\,1,\,2,\dots$ Their dimension (which we also call D^p for brevity) is then $D^p = p+1$:

$$D^0 = \mathbf{1}\,, \quad D^1 = \mathbf{2}\,, \quad D^2 = \mathbf{3}\,, \quad D^3 = \mathbf{4}\,, \quad \dots \tag{A.9}$$

In the two-dimensional fundamental representation D^1, the generators are the Pauli matrices ($\mathsf{t}_a = \tau_a/2$). Because $SU(2)$ has three generators, the adjoint representation is the three-dimensional one, D^2.

SU(3): The group $SU(3)$ has eight generators t_a. It has rank two and therefore there are two Casimir operators (namely $\mathsf{t}_a\,\mathsf{t}_a$ and $d_{abc}\,\mathsf{t}_a\,\mathsf{t}_b\,\mathsf{t}_c$) which label its irreducible representations. We call these representations D^{pq}; they depend on two quantum numbers $p,\,q=0,\,1,\,2,\dots$ and their dimension is

$$D^{pq} = \frac{1}{2}\,(p+1)(q+1)(p+q+2). \tag{A.10}$$

The lowest-dimensional irreducible representations are:

$$D^{00} = \mathbf{1}\,, \quad \begin{array}{l} D^{10} = \mathbf{3} \\ D^{01} = \overline{\mathbf{3}} \end{array}, \quad \begin{array}{l} D^{20} = \mathbf{6} \\ D^{02} = \overline{\mathbf{6}} \end{array}, \quad D^{11} = \mathbf{8}\,, \quad \begin{array}{l} D^{30} = \mathbf{10} \\ D^{03} = \overline{\mathbf{10}} \end{array}, \quad \dots \tag{A.11}$$

The fundamental triplet and antitriplet representations are D^{10} and D^{01}; the adjoint representation is the octet D^{11}; D^{00} is the singlet and D^{30} the decuplet. The multiplets can be constructed graphically as shown in Fig. A.1. The generators t_3 and $(2/\sqrt{3})\,\mathsf{t}_8$ commute with each other and form the Cartan subalgebra, so their eigenvalues I_3 and Y label the states within the multiplets which are therefore planar objects. The remaining generators are ladder operators and connect these states with each other:

$$\mathsf{t}_\pm = \mathsf{t}_1 \pm i\mathsf{t}_2\,, \qquad \mathsf{u}_\pm = \mathsf{t}_6 \pm i\mathsf{t}_7\,, \qquad \mathsf{v}_\pm = \mathsf{t}_4 \pm i\mathsf{t}_5\,. \tag{A.12}$$

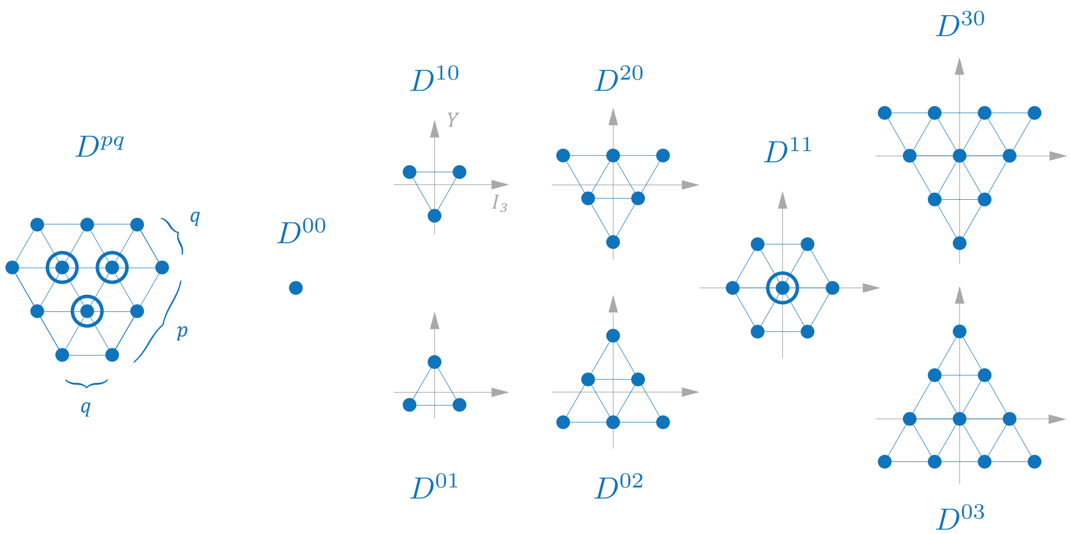


FIGURE A.1: $SU(3)$ multiplets. The left diagram shows the generic construction and the remaining ones correspond to the lowest-dimensional representations from Eq. (A.11).

A generic multiplet is a hexagon in the (I_3, Y) plane; for $p = 0$ or $q = 0$ the hexagons degenerate to triangles. Each hexagon includes further degenerate states that are obtained by lowering p and q by one unit each.

SU(4): The group $SU(4)$ has rank three and therefore we have three quantum numbers $p,\ q,\ r = 0,\ 1,\ 2, \dots$ to label the representations D^{prq}. Their dimensions are

$$D^{prq} = \frac{1}{12}\,(p+1)(q+1)(r+1)(p+r+2)(q+r+2)(p+q+r+3)\,. \qquad \text{(A.13)}$$

The lowest-dimensional irreducible representations are

$$D^{000} = \mathbf{1}\,, \quad \begin{array}{l} D^{100} = \mathbf{4} \\ D^{001} = \overline{\mathbf{4}} \end{array}, \quad D^{010} = \mathbf{8}\,, \quad \begin{array}{l} D^{200} = \mathbf{10} \\ D^{002} = \overline{\mathbf{10}} \end{array}, \quad D^{101} = \mathbf{15}\,, \quad \dots \qquad \text{(A.14)}$$

The fundamental representations are $\mathbf{4}$ and $\overline{\mathbf{4}}$, and $\mathbf{15}$ is the adjoint.

A.3 Product representations

Vectors that transform under the N-dimensional fundamental or antifundamental representations of $SU(N)$ satisfy the transformation law

$$\psi' = U\psi \quad \Leftrightarrow \quad \psi'_i = U_{ij}\,\psi_j\,, \qquad \psi'^{\dagger} = \psi^{\dagger}\,U^{\dagger} \quad \Leftrightarrow \quad \psi'^{\star}_i = U^{\star}_{ij}\,\psi^{\star}_j\,, \qquad \text{(A.15)}$$

where $U \in \{D^1, D^{10}, D^{100}, \dots\}$ and $U^\star \in \{D^1, D^{01}, D^{001}, \dots\}$. What happens if we take tensor products of ψ and $\psi^\star$? Higher-rank tensors are defined as those quantities that have the same transformation properties as the direct product of vectors. To keep track of the (anti-)fundamental nature of the representations where they originate from, it is helpful to introduce upper and lower indices and write:

$$\psi_i \to \psi^i\,, \qquad \psi_i^\star \to \psi_i\,, \qquad U_{ij} \to U^i{}_j\,, \qquad U_{ij}^\star \to U_i{}^j\,, \tag{A.16}$$

together with the Einstein summation convention (for example, $UU^\dagger = 1$ becomes $U^i{}_j\, U_k{}^j = \delta^i_k$). The coefficients of a generic $SU(N)$ tensor of rank (n,m) then transform under the product representation

$$\psi'^{\,i_1\dots i_n}{}_{j_1\dots j_m} = \big(U^{i_1}{}_{k_1}\dots U^{i_n}{}_{k_n}\big)\big(U_{j_1}{}^{l_1}\dots U_{j_m}{}^{l_m}\big)\,\psi^{k_1\dots k_n}_{l_1\dots l_m} \tag{A.17}$$

which is, however, not irreducible. To see this, permute the indices for example of a rank $(2,0)$ tensor — it commutes with the $SU(N)$ transformation:

$$\psi'^{\,ij} = U^i{}_k\, U^j{}_l\, \psi^{kl}\,, \qquad \psi'^{\,ji} = U^j{}_k\, U^i{}_l\, \psi^{kl} = U^i{}_k\, U^j{}_l\, \psi^{lk}\,. \tag{A.18}$$

Therefore, the symmetric and antisymmetric combinations

$$S^{ij} = \tfrac{1}{2}\left(\psi^{ij} + \psi^{ji}\right), \qquad A^{ij} = \tfrac{1}{2}\left(\psi^{ij} - \psi^{ji}\right) \tag{A.19}$$

do not mix under $SU(N)$ and form irreducible subspaces. A 2×2 matrix can be decomposed into one antisymmetric and three symmetric components, a 3×3 matrix has three antisymmetric and six symmetric components; so we can write[1]

$$SU(2):\quad \mathbf{2}\otimes\mathbf{2} = \mathbf{1}_A \oplus \mathbf{3}_S\,, \qquad SU(3):\quad \mathbf{3}\otimes\mathbf{3} = \overline{\mathbf{3}}_A \oplus \mathbf{6}_S\,. \tag{A.20}$$

These components transform now again under irreducible representations of $SU(N)$:

$$D^1 \otimes D^1 = D^0 \oplus D^2\,, \qquad D^{10} \otimes D^{10} = D^{01} \oplus D^{20}\,. \tag{A.21}$$

That is, if we arrange the components of the 2×2 tensor ψ^{ij} into a four-dimensional vector, the reducible representation matrix $D^1 \otimes D^1$ becomes block-diagonal:

$$\left(\begin{array}{c} \bullet \\ \hline \bullet \\ \bullet \\ \bullet \end{array}\right)' = \left(\begin{array}{c|c} D^0 & \\ \hline & D^2 \end{array}\right)\left(\begin{array}{c} \bullet \\ \hline \bullet \\ \bullet \\ \bullet \end{array}\right), \tag{A.22}$$

and similarly for the 3×3 matrix ψ^{ijk}.

While the symmetry argument is not directly applicable for tensors of mixed rank, the trace $\psi^i{}_i$ is invariant under $SU(N)$ and can be factored out. For example for a tensor of rank $(1,1)$:

$$\psi'^{\,i}{}_i = U^i{}_k\, U_i{}^l\, \psi^k{}_l = \delta^l_k\, \psi^k{}_l = \psi^i{}_i\,, \tag{A.23}$$

so we have $\mathbf{2}\otimes\overline{\mathbf{2}} = \mathbf{1}\oplus\mathbf{3}$ in $SU(2)$, $\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{1}\oplus\mathbf{8}$ in $SU(3)$, etc. Generally, in order to study product representations of $SU(N)$ one must work out the simultaneous irreducible representations of $SU(N)$ *and* the permutation group. The latter can be most easily obtained with the help of Young diagrams.

[1] It will become clear from the discussion of Young diagrams why $\overline{\mathbf{3}}$ instead of $\mathbf{3}$ appears here.

Young diagrams. Let's forget again about $SU(N)$ for the moment. Consider a quantity $f_{a_1\dots a_n}$ that carries n indices (where each can run from $1\dots N$), or suppose we want to form the $n-$fold tensor product of N-dimensional vectors: $\psi_{a_1}\otimes\dots\otimes\psi_{a_n}$. Without specifying the index range N, there are $n!$ possible permutations which fall into irreducible subspaces of the permutation group S_n. These can be visualized by Young diagrams: take n boxes and stick them together in all possible ways so that the number of boxes in each consecutive row (from top to bottom) and each consecutive column (from left to right) does not increase. For example:

$$S_2: \quad \ydiagram{2}\,,\ \ydiagram{1,1} \qquad S_3: \quad \ydiagram{3}\,,\ \ydiagram{2,1}\,,\ \ydiagram{1,1,1} \qquad S_4: \quad \ydiagram{4}\,,\ \ydiagram{3,1}\,,\ \ydiagram{2,2}\,,\ \ydiagram{2,1,1}\,,\ \ydiagram{1,1,1,1}\,.$$

Eventually we will fill these boxes with the N possible indices (or the N possible vector components), but let's see first how far we can get without doing that. A row denotes symmetrization, a column antisymmetrization. For S_3, $\ydiagram{3}$ is totally symmetric in all indices, $\ydiagram{2,1}$ has mixed symmetry, and the remaining diagram is totally antisymmetric. Each Young diagram corresponds to an irreducible representation of S_n.

The dimension d of a representation can be inferred from the number of 'standard tableaux' that it permits. A standard tableau is filled with (distinct) numbers from $1\dots n$ which increase in each row and each column. For example:

$$\underbrace{\begin{ytableau} 1 & 2 & 3 \end{ytableau}}_{d=1}\,, \qquad \underbrace{\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}}_{d=2}\,, \qquad \underbrace{\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau}}_{d=2}\,, \qquad \text{etc.} \tag{A.24}$$

For larger Young tableaux this exercise can become tedious. Fortunately, the dimension of a Young diagram can be also determined from the hook factor h which is the product of all hook lengths in a diagram. The hook length of a box counts the number of boxes directly below and to its right, plus counting the box itself. In the following Young tableau the hook lengths are given for each box:

$$\begin{ytableau} 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau} \quad \Rightarrow \quad h = 5\cdot 3\cdot 3\cdot 1\cdot 1\cdot 1 = 45\,. \tag{A.25}$$

The dimension of a Young diagram is then given by $d = n!/h$ (in this example, we would have $d = 6!/45 = 16$).

The dimension d is also the multiplicity of each diagram in the $n!$-dimensional reducible representation of S_n. For example in S_3, the 6 possible permutations of a function of three indices can be arranged in a symmetric singlet, an antisymmetric singlet, and two doublets:

$$\ydiagram{3} + 2\cdot\ydiagram{2,1} + \ydiagram{1,1,1} \quad \Rightarrow \quad n! = 6 = 1 + 2\cdot 2 + 1 = \sum_i d_i^2\,. \tag{A.26}$$

For S_2, we have $2 = 1+1$ ($\ydiagram{2}$ and $\ydiagram{1,1}$) and for S_4: $4! = 1 + 3\cdot 3 + 2\cdot 2 + 3\cdot 3 + 1$. Phrased differently: in S_2, we can arrange the 2 possible permutations of f_{ab} into a singlet and an antisinglet; in S_3, we can distribute the 6 permutations of f_{abc} into a singlet, an antisinglet and two doublets; and in S_4, we can arrange the 24 permutations of f_{abcd} into a singlet, an antisinglet, two doublets, three triplets and three antitriplets.

Combining $SU(N)$ and S_n. Let's start to fill up the Young diagrams. Suppose we are interested in the flavor wave functions of baryons in $SU(3)_F$. We have $N = 3$ flavors u, d and s at our disposal, and because we are dealing with baryons made of three quarks we consider the permutation group S_3. We start by writing down the possible flavor content of a wave function before symmetrization:

$$uuu, \quad uud, \quad udd, \quad ddd, \quad uus, \quad dds, \quad uss, \quad dss, \quad sss, \quad uds. \tag{A.27}$$

Each column in a Young diagram means antisymmetrization, and if we antisymmetrize the same flavor the result is zero. Therefore the flavor index must *increase* in vertical direction (in the sense $u \to d \to s$). On the other hand, each row implies symmetrization, and while we can of course symmetrize the same index ($\begin{array}{|c|c|}\hline u & u \\ \hline\end{array} \neq 0$), it doesn't matter in which direction we do it because the result will be the same (e.g., $\begin{array}{|c|c|}\hline u & d \\ \hline\end{array} = \begin{array}{|c|c|}\hline d & u \\ \hline\end{array}$). Hence, to avoid overcounting, the flavor index cannot become *smaller* in horizontal direction. If we stick to these simple rules, we obtain:

$$\begin{array}{lll}
uuu \longrightarrow \begin{array}{|c|c|c|}\hline u & u & u \\ \hline\end{array}\,, & \quad uud \longrightarrow \begin{array}{|c|c|c|}\hline u & u & d \\ \hline\end{array}\,, & \begin{array}{|c|c|}\hline u & u \\ \hline d \\ \cline{1-1}\end{array} \\[2ex]
ddd \longrightarrow \begin{array}{|c|c|c|}\hline d & d & d \\ \hline\end{array}\,, & \quad udd \longrightarrow \begin{array}{|c|c|c|}\hline u & d & d \\ \hline\end{array}\,, & \begin{array}{|c|c|}\hline u & d \\ \hline d \\ \cline{1-1}\end{array}
\end{array} \tag{A.28}$$

from which we can also read off how the remaining combinations work out: sss only yields a symmetric combination whereas uus, dds, uss and dss produce a singlet and a doublet each. Finally, the remaining uds produces all combinations:

$$uds \longrightarrow \begin{array}{|c|c|c|}\hline u & d & s \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline u & d \\ \hline s \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline u & s \\ \hline d \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|}\hline u \\ \hline d \\ \hline s \\ \hline\end{array}\,. \tag{A.29}$$

In summary, we arrived at one antisymmetric singlet, eight doublets, and 10 symmetric singlets. These multiplicities correspond to the irreducible representations of $SU(3)$:

$$D^{00}\ (\text{singlet}) \Leftrightarrow \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\,, \quad D^{11}\ (\text{octet}) \Leftrightarrow \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}\,, \quad D^{30}\ (\text{decuplet}) \Leftrightarrow \begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}\,.$$

In general, each irreducible representation of the group $SU(N)$ can be identified with a certain Young diagram. Consequently, a given diagram carries now a dimension d with respect to S_n, but also the dimension D of the $SU(N)$ representation. The latter can be determined by writing down all possible 'standard tableaux', where in contrast to (A.24) the entries can also be identical, so that they increase vertically and do not decrease horizontally. Similarly as before, one can determine the dimension D also via the hook factor: $D = F/h$, where F is determined as follows. Assign a factor N to the left upper box in the Young tableau; add 1 each time you go to the right and subtract 1 each time you go down:

$$\begin{array}{|c|c|c|}\hline N & N{+}1 & N{+}2 \\ \hline N{-}1 & N \\ \cline{1-2} N{-}2 \\ \cline{1-1}\end{array} \qquad \Rightarrow \qquad F = N^2 \cdot (N+1) \cdot (N+2) \cdot (N-1) \cdot (N-2)\ .$$

Conjugate representation. In terms of Young diagrams, the conjugate representations can be obtained in the following way. For each column, replace the j boxes in the column by $N-j$ boxes, and flip the diagram around the vertical axis. For example in $SU(4)$:

$$\overline{\ydiagram{1}} = \ydiagram{1,1,1}\,, \qquad \overline{\ydiagram{1,1}} = \ydiagram{1,1}\,, \qquad \overline{\ydiagram{2,1}} = \ydiagram{2,2,1}\,, \qquad \overline{\ydiagram{2,2}} = \ydiagram{2,2}\,. \tag{A.30}$$

This entails that representations which only differ by columns of length N attached to the left are equivalent, for example in $SU(3)$:

$$\overline{\ydiagram{2,1,1}} = \ydiagram{1,1} = \overline{\ydiagram{1}} \qquad \Rightarrow \qquad \ydiagram{2,1,1} = \ydiagram{1}\,, \tag{A.31}$$

or in $SU(2)$:

$$\ydiagram{2,1} = \ydiagram{1}\,, \qquad \ydiagram{3,1} = \ydiagram{2}\,, \qquad \ydiagram{3,3} = \ydiagram{2,2} = \ydiagram{1,1}\,. \tag{A.32}$$

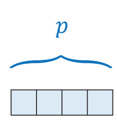
It also implies that in $SU(2)$ each conjugate representation is identical to the representation itself:

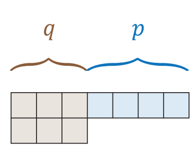
$$\overline{\ydiagram{1}} = \ydiagram{1}\,, \qquad \overline{\ydiagram{2}} = \ydiagram{2}\,, \qquad \overline{\ydiagram{3}} = \ydiagram{3}\,. \tag{A.33}$$

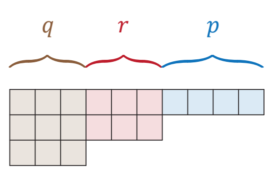
$SU(N)$ representations as Young diagrams. If we put everything together we arrive at Table A.2, which states the dimension D for the irreducible representations of $SU(N)$, together with their Young diagrams which carry dimension d. In this way we can identify each irreducible representation of $SU(N)$ directly with a Young diagram, for example for $SU(3)$:

$$D^{00} = \ydiagram{1,1,1}\,, \quad \begin{array}{l} D^{10} = \ydiagram{1} \\ D^{01} = \ydiagram{1,1} \end{array}, \quad \begin{array}{l} D^{20} = \ydiagram{2} \\ D^{02} = \ydiagram{2,2} \end{array}, \quad D^{11} = \ydiagram{2,1}\,, \quad \begin{array}{l} D^{30} = \ydiagram{3} \\ D^{03} = \ydiagram{3,3} \end{array}, \quad \ldots$$

In general, the correspondence is given by:

- $SU(2):\ D^p\ =$ 

- $SU(3):\ D^{pq}\ =$ 

- $SU(4):\ D^{prq}\ =$ 

In addition, arbitrarily many columns of length N can be attached from the left because this produces an equivalent representation.

	d	h	D	$SU(2)$	$SU(3)$	$SU(4)$
$\ydiagram{1}$	1	1	N	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$
$\left.\ydiagram{1,1,1}\right\} N-1$	1	$(N-1)!$	N	$\mathbf{2}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{4}}$
$\left.\ydiagram{1,1,1,1}\right\} N$	1	$N!$	1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\left.\ydiagram{2,1,1}\right\} N-1$	$N-1$	$\frac{N!}{N-1}$	N^2-1	$\mathbf{3}$	$\mathbf{8}$	$\mathbf{15}$
$\ydiagram{2}$	1	2	$\frac{N(N+1)}{2}$	$\mathbf{3}$	$\mathbf{6}$	$\mathbf{10}$
$\ydiagram{1,1}$	1	2	$\frac{N(N-1)}{2}$	$\mathbf{1}$	$\overline{\mathbf{3}}$	$\mathbf{6}$
$\ydiagram{3}$	1	6	$\frac{N(N+1)(N+2)}{6}$	$\mathbf{4}$	$\mathbf{10}$	$\mathbf{20}$
$\ydiagram{2,1}$	2	3	$\frac{N(N+1)(N-1)}{3}$	$\mathbf{2}$	$\mathbf{8}$	$\mathbf{20}$
$\ydiagram{1,1,1}$	1	6	$\frac{N(N-1)(N-2)}{6}$	-	$\mathbf{1}$	$\overline{\mathbf{4}}$
$\ydiagram{4}$	1	24	$\frac{N(N+1)(N+2)(N+3)}{24}$	$\mathbf{5}$	$\mathbf{15}$	$\mathbf{35}$
$\ydiagram{3,1}$	3	8	$\frac{N(N+1)(N+2)(N-1)}{8}$	$\mathbf{3}$	$\mathbf{15}$	$\mathbf{45}$
$\ydiagram{2,2}$	2	12	$\frac{N^2(N+1)(N-1)}{12}$	$\mathbf{1}$	$\overline{\mathbf{6}}$	$\mathbf{20}$
$\ydiagram{2,1,1}$	3	8	$\frac{N(N+1)(N-1)(N-2)}{8}$	-	$\mathbf{3}$	$\mathbf{15}$
$\ydiagram{1,1,1,1}$	1	24	$\frac{N(N-1)(N-2)(N-3)}{24}$	-	-	$\mathbf{1}$

Table A.2: Identification of irreducible $SU(N)$ representations with Young diagrams for S_2, S_3 and S_4. The second and third columns state the dimension d of the diagram and the hook factor h. The remaining columns show the dimension D of the $SU(N)$ representation, with examples for $SU(2)$, $SU(3)$ and $SU(4)$. The first four rows collect the fundamental, antifundamental, singlet and (highlighted in red) adjoint representations.

Product representations. Now let's return to the construction of product representations. So far we have only looked at products of fundamental representations, e.g.

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \left(\ydiagram{2} \oplus \ydiagram{1,1} \right) \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}\,, \tag{A.34}$$

which becomes $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}$ in $SU(3)$. Now suppose we want to evaluate a tensor product such as this:

$$\ydiagram{2,2,1} \otimes \ydiagram{2,1}\,. \tag{A.35}$$

How does the general rule work? Let's denote the left diagram by X and the right one by Y. The prescription goes like this: start by filling the boxes in the top row of Y with labels 'a', the boxes in the second row with labels 'b', etc. Take the 'a' boxes from the top row and attach them to X in all possible ways, so that the number of boxes in each consecutive row (from top to down) and each consecutive column (from left to right) does not increase. Then,

- If you end up with more than one a in a column, delete the diagram (because a column means antisymmetrization).
- If a diagram contains a column with N boxes, delete just that column (because it yields an equivalent representation). If it contains a column with more than N boxes, delete the diagram (because we cannot antisymmetrize more flavors than we have).

All identical diagrams count just once. Repeat these steps for the second row in Y with the b's, the third row with the c's and so on until you're done. One final step:

- In the resulting diagrams, go from right to left in the first row, then in the second row, etc. At any point along that path, the number of b boxes you have picked up must be smaller than the number of a boxes. If this is not the case, delete the diagram. (For example, if the top right box contains 'b', you can delete the diagram because the number of a's at this point is zero.) Apply the same logic for $\#c < \#b$, etc.

Here's an example:

$$\begin{aligned} \ydiagram{1} \otimes \ytableaushort{a,b} &= \left(\ytableaushort{{}a} \oplus \ytableaushort{{},a} \right) \otimes \ytableaushort{b} = \ytableaushort{{}ab} \oplus \ytableaushort{{}a,b} \oplus \ytableaushort{{}b,a} \oplus \ytableaushort{{},a,b} \\ &= \ytableaushort{{}a,b} \oplus \ytableaushort{{},a,b}\,. \end{aligned} \tag{A.36}$$

In the case of $SU(3)$, this becomes $\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{8}\oplus\mathbf{1}$. (For $SU(2)$, the same relation entails $\mathbf{2}\otimes\mathbf{1} = \mathbf{2}$ because the second diagram vanishes, and for $SU(4)$ we get $\mathbf{4}\otimes\mathbf{6} = \mathbf{20}\oplus\overline{\mathbf{4}}$.) Of course we would have obtained the same result faster if we had started from $\overline{\mathbf{3}}\otimes\mathbf{3}$, but with this strategy it is also straightforward to verify more complicated tensor products.

We collect some useful results:

- $SU(2)$: here the tensor representations can also be inferred from the angular momentum addition rules:

$$(2j+1)\otimes(2j'+1)=\bigoplus_{J=|j-j'|}^{j+j'}(2J+1) \quad \Rightarrow \quad \begin{aligned} \mathbf{2}\otimes\mathbf{2} &= \mathbf{1}\oplus\mathbf{3}\,, \\ \mathbf{3}\otimes\mathbf{3} &= \mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}\,, \\ \mathbf{4}\otimes\mathbf{4} &= \mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}\oplus\mathbf{7}\,, \\ \mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} &= \mathbf{2}\oplus\mathbf{2}\oplus\mathbf{4}\,. \end{aligned} \tag{A.37}$$

- $SU(3)$: conjugate representations are here no longer equivalent. Some frequently used decompositions are

$$\begin{aligned} \mathbf{3}\otimes\overline{\mathbf{3}} &= \mathbf{1}\oplus\mathbf{8}\,, & \mathbf{6}\otimes\overline{\mathbf{6}} &= \mathbf{1}\oplus\mathbf{8}\oplus\mathbf{27}\,, \\ \mathbf{3}\otimes\mathbf{3} &= \overline{\mathbf{3}}\oplus\mathbf{6}\,, & \mathbf{8}\otimes\mathbf{8} &= \mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27}\,, \\ \mathbf{3}\otimes\mathbf{6} &= \mathbf{8}\oplus\mathbf{10}\,, & \mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} &= \mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\,. \end{aligned} \tag{A.38}$$

- $SU(6)$: $\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}=\mathbf{20}\oplus\mathbf{70}\oplus\mathbf{70}\oplus\mathbf{56}$.