1 Classical scalar fields

Classical field theory. The action of a system described by classical mechanics is given by

$$S = \int_{t_1}^{t_2} dt\, L\,(q_i(t), \dot{q}_i(t)) = \int dt \left(\frac{1}{2} \sum_i \dot{q}_i^2 - V(q_1 \dots q_n) \right). \tag{1.1}$$

The transition to classical field theory proceeds via the replacements

$$q_i(t) \to \Phi(\boldsymbol{x}, t) \to \Phi(x)\,, \qquad \dot{q}_i(t) \to \frac{\partial \Phi(\boldsymbol{x}, t)}{\partial t} \to \partial_\mu \Phi(x)\,, \tag{1.2}$$

because in a relativistic theory the time derivative can only appear as a part of ∂_μ. The action then takes the form

$$S = \int dt\, L\big(\Phi(x), \partial_\mu \Phi(x)\big) = \int_V d^4x\, \mathcal{L}\big(\Phi(x), \partial_\mu \Phi(x)\big)\,, \tag{1.3}$$

where $\mathcal{L}$ is called the Lagrangian density or simply the **Lagrangian** of the theory.

To obtain the equations of motion, we vary the action with respect to Φ and $\partial_\mu \Phi$ in a given volume V with the boundary condition $\{\delta\Phi, \delta(\partial_\mu \Phi)\}\big|_{\partial V} = 0$. Hamilton's principle of stationary action $\delta S = 0$ then entails

$$\begin{aligned} 0 \stackrel{!}{=} \delta S &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi}\, \delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\, \delta(\partial_\mu \Phi) \right] \\ &= \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \delta\Phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\, \delta\Phi \right) \right], \end{aligned} \tag{1.4}$$

where we interchanged the variation with the derivative and performed a partial integration. The second bracket is a total derivative and can be converted to a surface integral via Gauss' law. It is zero because the field and its derivative vanish at the boundary:

$$\int_V d^4x\, \partial_\mu F^\mu = \int_{\partial V} d\sigma_\mu\, F^\mu = 0\,. \tag{1.5}$$

The remaining integrand must also vanish because $\delta\Phi$ is an arbitrary variation. This leads to the **Euler-Lagrange equations of motion**:

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0\,. \tag{1.6}$$

If the Lagrangian contains several fields $\Phi_i(x)$, one simply has to sum over them in Eq. (1.4) and the equations of motion hold for each component separately.

Finally, let's generalize the Hamiltonian formalism to the field-theoretical description. For a discrete system, the canonical conjugate momentum and Hamilton function are given by

$$p_i(t) = \frac{\partial L}{d\dot{q}_i(t)}\,, \qquad H = \sum_i \dot{q}_i\, p_i - L\,. \tag{1.7}$$

In the continuum limit, the conjugate momentum becomes the canonically conjugate momentum *density* $\Pi(x)$,

$$p(\boldsymbol{x},t) = \frac{\partial L}{\partial\dot\Phi(\boldsymbol{x},t)} \quad \to \quad \Pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\Phi(x)}\,, \tag{1.8}$$

and the Hamilton function acquires the form

$$H = \int d^3x\,\Big(\Pi(x)\,\dot\Phi(x) - \mathcal{L}\Big) =: \int d^3x\,\mathcal{H}(x)\,, \tag{1.9}$$

where $\mathcal{H}(x)$ is the Hamiltonian density.

Real scalar field and Klein-Gordon equation. We start with the simplest example of a field theory. It contains only one type of field: a real scalar field $\Phi(x) = \Phi^*(x)$. What are the possible terms that can appear in the Lagrangian? $\mathcal{L}$ must be a Lorentz scalar, so it can only depend on Φ and $\partial_\mu\Phi\,\partial^\mu\Phi$ (and higher powers of these expressions). The combination $\partial_\mu\partial^\mu\Phi = \Box\,\Phi$ is a total derivative, so it doesn't change the equations of motion. Based on these considerations we write

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\Phi\,\partial^\mu\Phi - \frac{1}{2}\,m^2\Phi^2 - V\big(\Phi^n, \Phi^n(\partial\Phi)^m\big). \tag{1.10}$$

The first two terms define the Lagrangian for a free scalar field, whereas the potential V contains higher possible interaction terms.[1] The action is $S = \int d^4x\,\mathcal{L}$, and we can check that the mass dimensions work out correctly:

$$[S] = 0, \qquad [d^4x] = -4, \qquad [\mathcal{L}] = 4, \qquad [\Phi] = 1, \qquad [\partial_\mu] = 1, \tag{1.11}$$

and therefore the parameter m has indeed the dimension of a mass. Discarding the interaction terms (which we will always do in this chapter, hence 'free fields'), we can easily work out the Euler-Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\Phi} = -m^2\Phi\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = \partial^\mu\Phi\,, \qquad \partial_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = \partial_\mu\partial^\mu\Phi = \Box\Phi\,, \tag{1.12}$$

and thereby arrive at the **Klein-Gordon equation:**

$$(\Box + m^2)\Phi = 0\,. \tag{1.13}$$

To derive the Hamiltonian density, we have to find the conjugate momentum:

$$\mathcal{L} = \frac{1}{2}\Big(\dot\Phi^2 - (\nabla\Phi)^2 - m^2\Phi^2\Big) \quad \Rightarrow \quad \Pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\Phi(x)} = \dot\Phi(x)\,, \tag{1.14}$$

and therefore we obtain

$$\mathcal{H} = \Pi\,\dot\Phi - \mathcal{L} = \Pi^2 - \mathcal{L} = \frac{1}{2}\left(\Pi^2 + (\nabla\Phi)^2 + m^2\Phi^2\right). \tag{1.15}$$

[1] In the quantum field theory, renormalizability will limit their form to Φ^3 and Φ^4 interactions.

The solutions of the Klein-Gordon equation are plane waves $e^{\pm ipx}$ with dispersion relation $p^2 = m^2 \Rightarrow p_0 = \pm\sqrt{\boldsymbol{p}^2 + m^2} = \pm E_p$, so we can write its general solutions as

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \left(a(\boldsymbol{p})\, e^{-ipx} + a^*(\boldsymbol{p})\, e^{ipx} \right) \Big|_{p^0=E_p} . \tag{1.16}$$

The overall normalization with $(2\pi)^{-3/2}$ and the factor $2E_p$ in the integral measure are just a matter of convention at this point, because we could equally absorb them into the Fourier coefficients $a(\boldsymbol{p})$ and $a^*(\boldsymbol{p})$. Later we will find that $d^3p/(2E_p)$ defines a Lorentz-invariant integral measure, so we keep it for convenience. Furthermore, setting $p^0 = +E_p$ does not restrict us to positive-energy solutions because we would get the same form with $p^0 = -E_p$ except for the interchange $a(\boldsymbol{p}) \leftrightarrow a^*(-\boldsymbol{p})$, which we can always redefine (to see this, replace $\boldsymbol{p} \to -\boldsymbol{p}$ as integration variable). The interpretation of the positive- and negative-frequency modes $e^{\mp ipx}$ will become clear only after quantizing the theory.

Complex scalar field. We can generalize the formalism to complex scalar fields:

$$\Phi(x) = \frac{1}{\sqrt{2}} \left(\Phi_1(x) + i\Phi_2(x) \right) , \qquad \Phi_i^*(x) = \Phi_i(x) , \tag{1.17}$$

whose Lagrangian can be written as the superposition of the Lagrangians for its real and imaginary parts:

$$\mathcal{L} = \sum_{i=1}^{2} \left[\frac{1}{2}\, \partial_\mu \Phi_i\, \partial^\mu \Phi_i - \frac{m^2}{2} \Phi_i^2 \right] = \partial_\mu \Phi^*\, \partial^\mu \Phi - m^2\, |\Phi|^2 . \tag{1.18}$$

If we view the fields $\Phi(x)$ and $\Phi^*(x)$ as the independent degrees of freedom, the conjugate momenta become

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}(x)} = \dot{\Phi}^*(x) , \qquad \Pi^*(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^*(x)} = \dot{\Phi}(x) \tag{1.19}$$

and the Hamiltonian is

$$H = \int d^3x \left(\Pi^* \dot{\Phi}^* + \Pi\, \dot{\Phi} - \mathcal{L} \right) = \int d^3x \left(|\Pi|^2 + |\nabla\Phi|^2 + m^2\, |\Phi|^2 \right) . \tag{1.20}$$

Both fields satisfy Klein-Gordon equations: $(\Box + m^2)\, \Phi = (\Box + m^2)\, \Phi^* = 0$, and the Fourier expansion for their solutions has now the form

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \left(a(\boldsymbol{p})\, e^{-ipx} + b(\boldsymbol{p})^*\, e^{ipx} \right) \Big|_{p^0=E_p} , \tag{1.21}$$

with two independent coefficients $a(\boldsymbol{p})$ and $b(\boldsymbol{p})$.

We can define a Lorentz-invariant scalar product for solutions of the Klein-Gordon equation: **(Ex)**

$$\langle \Phi, \Psi \rangle := i \int d\sigma_\mu\, \Phi^*(x)\, \overset{\leftrightarrow}{\partial^\mu} \psi(x) = i \int d^3x\, \Phi^*(x)\, \overset{\leftrightarrow}{\partial_0}\, \psi(x) , \tag{1.22}$$

where $f\overset{\leftrightarrow}{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f)g$ and σ is a spacelike hypersurface (which we chose to be a fixed timeslice in the second step). The scalar product is Lorentz-invariant and therefore it has the same value on each spacelike hypersurface:

$$\left[\int_{\sigma_2} - \int_{\sigma_1}\right] d\sigma_\mu \, \Phi^* \overset{\leftrightarrow}{\partial^\mu} \Psi = \int d^4x \, \partial_\mu \, (\Phi^* \overset{\leftrightarrow}{\partial^\mu} \Psi) = \int d^4x \, (\Phi^* \, \Box\Psi - \Box\Phi^* \, \Psi) = 0 \,. \tag{1.23}$$

In the first step we used Gauss' law under the assumption that the fields vanish sufficiently fast at $|\boldsymbol{x}| \to \infty$, and to obtain the zero we inserted the Klein-Gordon equations for the fields Φ and Ψ. Hence, although the fields are time-dependent, the second form in Eq. (1.22) is independent of time. Eq. (1.22) is linear in the second argument and antilinear in the first, it satisfies $\langle \Phi, \Psi \rangle^* = \langle \Psi, \Phi \rangle$, but it is not positive definite: to see this, consider the plane waves

$$f_p(x) = \frac{1}{(2\pi)^{3/2}} \, e^{-ipx}\Big|_{p^0 = E_p} \qquad \Rightarrow \qquad \begin{aligned} \langle f_p, f_{p'} \rangle &= 2E_p \, \delta^3(\boldsymbol{p} - \boldsymbol{p}') \,, \\ \langle f_p^*, f_{p'}^* \rangle &= -2E_p \, \delta^3(\boldsymbol{p} - \boldsymbol{p}') \,, \\ \langle f_p, f_{p'}^* \rangle &= 0 \,. \end{aligned} \tag{1.24}$$

With their help we can write the free Klein-Gordon solutions (1.21) as ($a_p = a(\boldsymbol{p})$)

$$\Phi(x) = \int \frac{d^3p}{2E_p} \left(a_p \, f_p(x) + b_p^* \, f_p^*(x) \right) , \tag{1.25}$$

and therefore

$$\langle \Phi, \Phi \rangle = \int \frac{d^3p}{2E_p} \int \frac{d^3p'}{2E_{p'}} \langle a_p f_p + b_p^* f_p^*, a_{p'} f_{p'} + b_{p'}^* f_{p'}^* \rangle = \int \frac{d^3p}{2E_p} \left(|a_p|^2 - |b_p|^2 \right) . \tag{1.26}$$

The norm is not positive definite because of the negative-energy contributions $|b_p|^2$, hence it does not permit a probability interpretation. Later we will see that $\langle \Phi | \Phi \rangle$ coincides with the $U(1)$ charge for a complex scalar field. For a real scalar field it is zero because $b_p = a_p$. From Eqs. (1.24–1.25) we can extract the Fourier coefficients via

$$a_p = \langle f_p, \Phi \rangle, \qquad b_p^* = -\langle f_p^*, \Phi \rangle \,. \tag{1.27}$$

Noether theorem. Symmetries play a fundamental role in field theories. For example, Poincaré invariance was the guiding principle for the construction of the Lagrangian (1.10), and eventually we will see that also the properties of 'mass' and 'spin' of a particle have their origin in the Poincaré group (they are related to the Casimir operators of the group). There are also other types of symmetries such as *internal* symmetries, and generally the invariance of the action under a symmetry leads to conserved currents and charges. Symmetries also have dynamical implications: in fact, the very nature of the Standard Model as a collection of gauge theories, where charged particles interact via gauge bosons, is a consequence of *gauge invariance*.

Consider a field theory with fields $\Phi_i(x)$ and action S. We perform a transformation of the coordinates and fields, which are parametrized by infinitesimal parameters ε_a:

$$\begin{aligned} x'^\mu &= x^\mu + \delta x^\mu \,, & \delta x^\mu &= \textstyle\sum_a \varepsilon_a \, X_a^\mu(x) \,, \\ \Phi_i'(x') &= \Phi_i(x) + \delta\Phi_i \,, & \delta\Phi_i &= \textstyle\sum_a \varepsilon_a \, F_{ia}(\Phi, \partial\Phi) \,. \end{aligned} \tag{1.28}$$

The **Noether theorem** states that for each transformation that leaves the action invariant (then we call it a *symmetry transformation*) there is a conserved Noether current $j_a^\mu(x)$ with

$$\partial_\mu \, j_a^\mu(x) = 0 \tag{1.29}$$

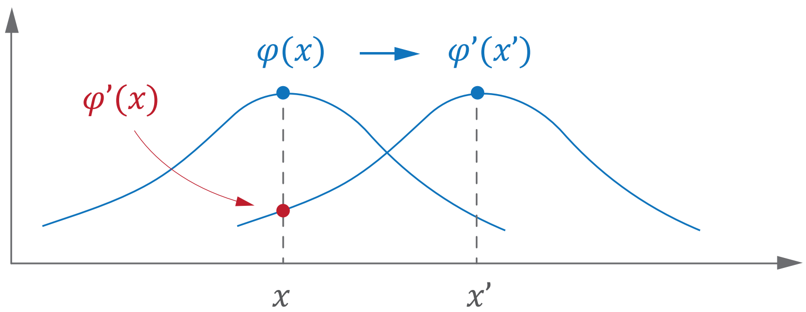

FIGURE 1.1: Visualization of Eqs. (1.33–1.34).

along the classical trajectories, i.e., for solutions of the classical equations of motion. We note that one can still write down a Noether current $j_a^\mu(x)$ irrespective of whether the transformation (1.28) is a symmetry or not (then it won't be conserved), and in general we do not require the fields $\Phi_i(x)$ to satisfy the classical equations of motion.

Here are some examples for symmetry transformations:

- **Internal symmetries** correspond to transformations of the fields only, but not spacetime itself. They are usually realized in the form of Lie groups whose elements are obtained by exponentiating the group generators G_a:

$$\Phi_i'(x) = D_{ij}\,\Phi_j(x)\,, \quad D = e^{i\sum\limits_a \varepsilon_a G_a} \quad \Leftrightarrow \quad \begin{aligned} \delta x^\mu &= 0\,, \\ \delta\Phi_i &= i\sum_a \varepsilon_a\,(G_a)_{ij}\,\Phi_j\,. \end{aligned} \tag{1.30}$$

- Spacetime **translations** depend on four parameters a^μ and they are part of the Poincaré group:

$$\begin{aligned} x' &= x + a \\ \Phi_i'(x+a) &= \Phi_i(x) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \delta x^\mu &= a^\mu \\ \delta\Phi_i &= 0\,. \end{aligned} \tag{1.31}$$

- **Lorentz transformations** consist of rotations and boosts and contain the remaining six parameters of the Poincaré group. An infinitesimal Lorentz transformation $\Lambda = 1 + \varepsilon$ is parametrized by the antisymmetric matrix $\varepsilon_{\mu\nu}$:

$$\begin{aligned} x' &= \Lambda x \\ \Phi_i'(\Lambda x) &= D_{ij}(\Lambda)\,\Phi_j(x) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \delta x^\mu &= \varepsilon^{\mu\nu} x_\nu \\ \delta\Phi_i &= \dots \end{aligned} \tag{1.32}$$

  The matrices $D(\Lambda)$ are the finite-dimensional irreducible representations of the Lorentz group which depend on the nature of the fields (scalar, Dirac, vector field etc.); we will discuss them later in the context of Dirac theory. For scalar fields, $D(\Lambda) = 1$ and therefore they satisfy $\Phi_i'(\Lambda x) = \Phi_i(x)$ and $\delta\Phi_i = 0$ (which is why the fields are *scalars* under Lorentz transformations).

To proceed, we need to define two types of variations. The 'total' variation is what we already introduced above:

$$\delta\Phi_i = \Phi_i'(x') - \Phi_i(x)\,. \tag{1.33}$$

It vanishes for the example of a scalar field under Poincaré transformations. The second type of variation is the change of the *functional form* of the field at the position x:

$$\begin{aligned}\delta_0\Phi_i &= \Phi_i'(x) - \Phi_i(x) \\ &= \Phi_i'(x'-\delta x) - \Phi_i(x) = \Phi_i'(x') - \partial_\mu\Phi_i\,\delta x^\mu - \Phi_i(x) \\ &= \delta\Phi_i - \partial_\mu\Phi_i\,\delta x^\mu\,.\end{aligned} \tag{1.34}$$

Both types of variations are visualized in Fig. 1.1: a scalar field is invariant under translations and therefore $\Phi'(x') = \Phi(x)$; however, the functional form $\Phi'(x)$ at the position x has changed in the process. It follows that

$$\delta\Phi_i = \delta_0\Phi_i + \partial_\mu\Phi_i\,\delta x^\mu\,, \tag{1.35}$$

where the second term vanishes for internal symmetries ($\delta x^\mu = 0$).

Consider now a variation of the action of the form

$$\begin{aligned}\delta S &= \int d^4x'\,\mathcal{L}\big(\Phi'(x'), \partial_\mu'\Phi'(x')\big) - \int d^4x\,\mathcal{L}\big(\Phi(x), \partial_\mu\Phi(x)\big) \\ &= \int d^4x\,\delta\mathcal{L} + \int (\delta\,d^4x)\,\mathcal{L}\,,\end{aligned} \tag{1.36}$$

with $\partial_\mu' = \partial/\partial x'^\mu$, which does not vanish at the boundary and also permits a variation of the volume itself. The variation of the integral measure follows from expanding the Jacobian of the transformation:

$$d^4x' = |\det J|\,d^4x = (1 + \partial_\mu\delta x^\mu + \dots)\,d^4x \quad\Rightarrow\quad \delta\,d^4x = (\partial_\mu\delta x^\mu)\,d^4x\,. \tag{1.37}$$

Inserting this together with Eq. (1.35) into the expression for δS, we get

$$\begin{aligned}\delta S &= \int d^4x\,[\,\delta_0\mathcal{L} + \partial_\mu\mathcal{L}\,\delta x^\mu + \mathcal{L}\,\partial_\mu\delta x^\mu\,] \\ &= \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\Phi}\,\delta_0\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\delta_0\,\partial_\mu\Phi + \partial_\mu(\mathcal{L}\,\delta x^\mu)\right] \\ &= \int_V d^4x\Bigg\{\underbrace{\left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right]}_{\text{eqs. of motion}}\delta_0\Phi + \partial_\mu\underbrace{\left[\mathcal{L}\,\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\delta_0\Phi\right]}_{-\delta j^\mu = -\sum_a \varepsilon_a\,j_a^\mu}\Bigg\}\,.\end{aligned} \tag{1.38}$$

In the second bracket we defined a current δj^μ; it inherits the dependence on the infinitesimal transformation parameters ε_a in Eq. (1.28), so there is one current j_a^μ for each parameter ε_a. Now, if these transformations define a symmetry of the action then $\delta S = 0$, and because the spacetime volume is arbitrary also the integrand must be zero. The first bracket vanishes upon inserting the solutions of the classical equations of motion, and so we arrive at a conserved Noether current for each ε_a:

$$\partial_\mu \, j_a^\mu(x) = 0\,. \tag{1.39}$$

We can rewrite the Noether current in a more useful form. With Eq. (1.35) we eliminate $\delta_0\Phi$ in favor of the total variation $\delta\Phi$:

$$-\,\delta j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\delta\Phi - \underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\partial^\nu\Phi - g^{\mu\nu}\mathcal{L}\right]}_{=:\,T^{\mu\nu}}\delta x_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\delta\Phi - T^{\mu\nu}\delta x_\nu\,. \tag{1.40}$$

$T^{\mu\nu}$ is the **energy-momentum tensor** whose meaning will become clear in a moment. While we derived the Noether theorem and the current for a single-component field, the derivation goes through for arbitrary types of fields in arbitrary representations of the Lorentz group — one simply has to sum over all fields in the Lagrangian. Let's exemplify the case of a scalar field $\Phi(x)$ under ...

- translations ($\delta\Phi = 0$, $\delta x_\nu = a_\nu$): the first term in δj^μ vanishes, and after removing the translation parameters a^ν we find that the conserved current according to translation invariance is the energy-momentum tensor itself. The continuity equation
$$\partial_\mu \, T^{\mu\nu} = 0 \tag{1.41}$$
holds for solutions of the Klein-Gordon equation and can be easily verified. The energy-momentum tensor has the form $T^{\mu\nu} = \partial^\mu\Phi\,\partial^\nu\Phi - g^{\mu\nu}\,\mathcal{L}$, which corresponds to one current for each component of a^ν.

- Lorentz transformations ($\delta\Phi = 0$, $\delta x_\alpha = \varepsilon_{\alpha\beta}\,x^\beta$): here we can exploit the antisymmetry of $\varepsilon_{\alpha\beta}$ and write
$$-\,\delta j^\mu = -T^{\mu\alpha}\varepsilon_{\alpha\beta}\,x^\beta = -\frac{\varepsilon_{\alpha\beta}}{2}\left(T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha\right) =: -\frac{\varepsilon_{\alpha\beta}}{2}\,m^{\mu,\alpha\beta}\,. \tag{1.42}$$
Therefore, the conserved current is the **angular momentum density**
$$m^{\mu,\alpha\beta} = T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha\,, \qquad \partial_\mu\, m^{\mu,\alpha\beta} = 0\,. \tag{1.43}$$
It carries the orbital angular momentum of the field; for fields with higher spin there will be additional spin contributions coming from the $\delta\Phi$ term in Eq. (1.40). We can make this more explicit by inserting the energy-momentum tensor:
$$m^{\mu,\alpha\beta} = -i\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\underbrace{\left[-i\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right)\right]}_{=:\;L^{\alpha\beta}}\,\Phi + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha})\mathcal{L}\,. \tag{1.44}$$
$L^{\alpha\beta}$ is a 'covariantized' version of the orbital angular momentum, because in analogy to Eq. (2.55) we can define a three-vector
$$L^i := -\tfrac{1}{2}\,\varepsilon_{ijk}\,L^{jk} = i\varepsilon_{ijk}\,x^j\partial^k \quad\Rightarrow\quad \boldsymbol{L} = \boldsymbol{x}\times(-i\boldsymbol{\nabla})\,. \tag{1.45}$$
For a scalar field the current has the explicit form
$$m^{\mu,\alpha\beta} = -i\,\partial^\mu\Phi\,L^{\alpha\beta}\Phi + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha})\,\mathcal{L}\,. \tag{1.46}$$

- Internal symmetries ($\delta x_\nu = 0$): in that case only the first term in Eq. (1.40) contributes. An example is the $U(1)$ current of a complex scalar field that we will discuss in Eq. (2.44). The Lagrangian is invariant under the transformation $\Phi' = e^{i\varepsilon}\Phi$, $\Phi'^* = e^{-i\varepsilon}\Phi^*$, with ε a real constant, and the corresponding current is

$$j^\mu = i(\Phi^*\,\partial^\mu\Phi - \partial^\mu\Phi^*\,\Phi) = i\,\Phi^*\overset{\leftrightarrow}{\partial^\mu}\Phi\,, \qquad \partial_\mu\,j^\mu = 0\,. \tag{1.47}$$

Noether charges. There is another important consequence of current conservation. After inserting the equations of motion into Eq. (1.38), we can exploit Gauss' law to convert the remaining volume integral into a surface integral:

$$0 = \int\limits_V d^4x\,\partial_\mu\,j_a^\mu = \int\limits_{\partial V} d\sigma_\mu\,j_a^\mu\,. \tag{1.48}$$

Specifically, if we squeeze the spacetime volume between two hypersurfaces at fixed times, $d\sigma_\mu = (d^3x, \mathbf{0})$, and assume that the fields vanish sufficiently fast for $|\boldsymbol{x}| \to \infty$, we conclude that there is a conserved charge that has the same value for all times:

$$\int d^3x\,j_a^0\,\Big|_{t_2} - \int d^3x\,j_a^0\,\Big|_{t_1} = 0 \quad\Rightarrow\quad Q_a := \int d^3x\,j_a^0(x) = \text{const} \quad \forall\,t\,. \tag{1.49}$$

The relative sign comes from the fact that the normal vectors for the planes at fixed times always point outwards of the volume. In fact, this relation holds for arbitrary spacelike hypersurfaces, so the charge Q_a has the same value for all spacelike surfaces.

As an example, let's consider again a scalar field under translations. The conserved current is $T^{\mu\nu}$, and therefore the conserved charges are the spatial integrals $\int d^3x\,T^{0\nu}$. They coincide with the Hamiltonian H and the total momentum $\boldsymbol{P}$ of the system, which are conserved:

$$\begin{aligned} H &= \int d^3x\,T^{00} = \int d^3x\,(\Pi\,\dot\Phi - \mathcal{L}) = \int d^3x\,\mathcal{H}\,, \\ P^i &= \int d^3x\,T^{0i} = \int d^3x\,\Pi\,\partial^i\Phi = -\int d^3x\,\Pi\,\nabla_i\Phi\,. \end{aligned} \tag{1.50}$$

Taken together with the Lorentz transformations, the conserved charges are the quantities

$$P^\alpha = \int d^3x\,T^{0\alpha}\,, \qquad M^{\alpha\beta} = \int d^3x\,m^{0,\alpha\beta}\,. \tag{1.51}$$

For example, in the case of rotations the conserved charge is

$$J^i = -\tfrac{1}{2}\,\varepsilon_{ijk}\,M^{jk} = -\tfrac{1}{2}\,\varepsilon_{ijk}\int d^3x\,m^{0,jk} \overset{(1.45)}{=} \tfrac{i}{2}\,\varepsilon_{ijk}\int d^3x\,\Pi\,L^{jk}\,\Phi\,, \tag{1.52}$$

which is the total angular momentum of the field:

$$\boldsymbol{J} = -i\int d^3x\,\Pi\,\boldsymbol{L}\,\Phi\,. \tag{1.53}$$

Another example is the conserved charge for the $U(1)$ current in Eq. (1.47):

$$Q = i \int d^3x \, \Phi^* \overset{\leftrightarrow}{\partial_0} \Phi \,. \tag{1.54}$$

This is just our earlier construction of the 'norm' for Klein-Gordon solutions; it is indeed Lorentz-invariant because it has the same value on each spacelike hypersurface.

The Noether charges will play a prominent role in the quantum field theory. After quantizing the fields by imposing commutator relations, the charges inherit the operator structure of the fields and form a representation of the Lie algebra of the symmetry group on the Fock space. That is, if the group elements of the symmetry transformation in Eq. (1.28) can be written as

$$D = e^{i \sum\limits_a \varepsilon_a G_a} \qquad \text{with} \qquad [G_a, G_b] = i f_{abc} \, G_c \,, \tag{1.55}$$

with some generic structure constants f_{abc}, then the charges will satisfy the same Lie-algebra relation as the generators:

$$[Q_a, Q_b] = i f_{abc} \, Q_c \,, \tag{1.56}$$

and thereby provide a representation of the symmetry group on the state space. This is also true for the Poincaré group (which is also a Lie group): after quantization, the *operators* $M^{\mu\nu}$ and P^μ in Eq. (1.51) satisfy the commutator relations of the Poincaré algebra and thereby form a unitary representation of the Poincaré group on the state space.