11 Renormalization of QED

Renormalization constants. We have already discussed the underlying principles of renormalization in the context of a scalar field. This saves us from the trouble of going through the same steps all over again: we don't need to calculate loop diagrams in QED only to realize that they diverge and then figure out what to do about it, because with a few adaptations we can take over the ideas from the scalar theory.

Once again, we interpret all quantities in the Lagrangian as bare und unphysical,

$$\begin{aligned}\mathcal{L}_{\text{QED}} \stackrel{\text{p.I.}}{\simeq} \;& \bar{\psi}_{\text{B}}\,(i\partial\!\!\!/ - m_{\text{B}})\,\psi_{\text{B}} + g_{\text{B}}\,\bar{\psi}_{\text{B}}\,A\!\!\!/_{\text{B}}\,\psi_{\text{B}} \\ & + \frac{1}{2}A^\mu_{\text{B}}\,(\Box\, g_{\mu\nu} - \partial_\mu\partial_\nu)\,A^\nu_{\text{B}} + \frac{\lambda_{\text{B}}}{2}\,A^\mu_{\text{B}}\,\partial_\mu\,\partial_\nu A^\nu_{\text{B}}\,,\end{aligned} \tag{11.1}$$

and define their renormalized counterparts by

$$\psi_{\text{B}} = Z_\psi^{1/2}\,\psi\,, \quad A_{\text{B}} = Z_A^{1/2} A\,, \quad m_{\text{B}} = Z_m\, m\,, \quad g_{\text{B}} = Z_g\, g\,, \quad \lambda_{\text{B}} = Z_\lambda\,\lambda\,. \tag{11.2}$$

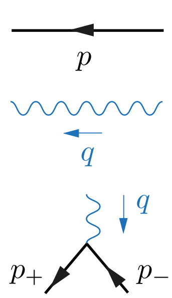
In principle there are five renormalization constants, but we will later see that gauge invariance relates two of them via Ward identities:

$$Z_g\, Z_A^{1/2} = 1\,, \qquad Z_\lambda\, Z_A = 1\,. \tag{11.3}$$

Hence, there are just three independent renormalization constants: Z_ψ, Z_A and Z_m. The resulting Lagrangian takes the form[8]

$$\begin{aligned}\mathcal{L}_{\text{QED}} = \;& Z_\psi\,\bar{\psi}\,(i\partial\!\!\!/ - Z_m\, m)\,\psi + Z_\psi\, g\,\bar{\psi} A\!\!\!/\,\psi \\ & + Z_A\,\frac{1}{2}A^\mu\,(\Box\, g_{\mu\nu} - \partial_\mu\partial_\nu)\,A^\nu + \frac{\lambda}{2}\,A^\mu\,\partial_\mu\,\partial_\nu A^\nu\,.\end{aligned} \tag{11.4}$$

The price we have to pay is that the renormalization constants now also enter in the Feynman rules:



$$\begin{aligned} S_0(p) &= \frac{i}{Z_\psi}\,\frac{p\!\!\!/ + m_{\text{B}}}{p^2 - m_{\text{B}}^2 + i\epsilon}\,, \\ D_0^{\mu\nu}(q) &= -\frac{i}{q^2 + i\epsilon}\left(\frac{1}{Z_A}\,T_q^{\mu\nu} + \frac{1}{\lambda}\,L_q^{\mu\nu}\right), \\ ig\,\Gamma_0^\mu(p,q) &= ig\,Z_\psi \gamma^\mu\,. \end{aligned} \tag{11.5}$$

Note that we pulled out a factor ig in defining the vertex Γ_0^μ. Our momentum routing for the vertex is such that the photon momentum is $q = p_f - p_i$ and the average fermion

[8]If $Z_\Gamma = Z_\psi\, Z_g\, Z_A^{1/2}$ denotes the prefactor that we would get in front of the coupling term $\sim \bar{\psi} A\!\!\!/\,\psi$, then the first condition in Eq. (11.3) is equivalent to $Z_\Gamma = Z_\psi$. To compare with the standard notation in the literature, set $Z_\Gamma = Z_1$, $Z_\psi = Z_2$ and $Z_A = Z_3$.

momentum is $p = (p_f + p_i)/2$. Along the same lines as earlier (using the projectors $T_q^{\mu\nu} = g^{\mu\nu} - q^\mu q^\nu/q^2$ and $L_q^{\mu\nu} = q^\mu q^\nu/q^2$) we obtain the inverse tree-level propagators:

$$iS_0^{-1}(p) = Z_\psi\,(\not{p} - m_\mathrm{B})\,, \qquad i(D_0^{-1})^{\mu\nu}(q) = -q^2\,(Z_A\,T_q^{\mu\nu} + \lambda\,L_q^{\mu\nu})\,. \tag{11.6}$$

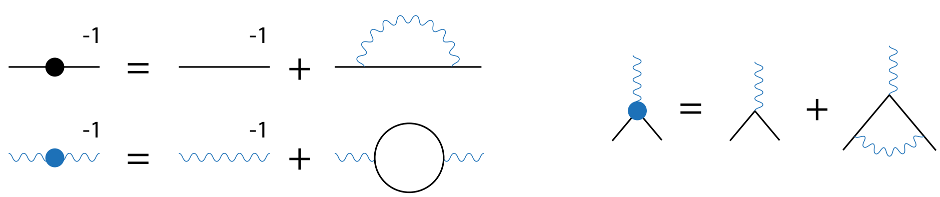
In analogy to the scalar theory, we can get the full 1PI Green functions (the inverse propagators and the fermion-photon vertex) by resumming its 1PI loop contributions. Omitting momentum arguments, this means for the fermion propagator

$$\begin{aligned} S &= S_0 + S_0\,i\Sigma\,S_0 + S_0\,i\Sigma\,S_0\,i\Sigma\,S_0 + \dots \\ &= S_0\,[1 + i\Sigma\,(S_0 + S_0\,i\Sigma\,S_0 + \dots)] = S_0\,(1 + i\Sigma\,S) \\ \Rightarrow\ S^{-1} &= S_0^{-1} - i\Sigma \quad \text{or} \quad iS^{-1} = iS_0^{-1} + \Sigma\,. \end{aligned} \tag{11.7}$$

$\Sigma(p)$ is the fermion self-energy, the sum of all 1PI loop contributions to the propagator. Applying the same steps to the photon propagator, we arrive at the perturbative series for the inverse propagators and the vertex:

$$\begin{aligned} iS^{-1}(p) &= iS_0^{-1}(p) + \Sigma(p)\,, \\ i(D^{-1})^{\mu\nu}(q) &= i(D_0^{-1})^{\mu\nu}(q) + \Pi^{\mu\nu}(q)\,, \\ \Gamma^\mu(p,q) &= \Gamma_0^\mu(p,q) + \Omega^\mu(p,q)\,. \end{aligned} \tag{11.8}$$

The terms on the right-hand side define the fermion **self-energy** $\Sigma(p)$, the photon **vacuum polarization** $\Pi^{\mu\nu}(q)$, and the **vertex correction** $\Omega^\mu(p,q)$. To lowest order in perturbation theory they are given by the following one-loop diagrams:



Tensor decomposition. Before we proceed, let's pause for a moment and think about the general tensor decomposition of these quantities. The self-energy depends on one momentum p, so the only possible tensor structures compatible with Lorentz covariance are $\not{p}$ and $\mathbb{1}$ (γ_5 or $\gamma_5\,\not{p}$ would have the wrong sign under a parity transformation), and the coefficients can only depend on the Lorentz-invariant p^2:

$$\Sigma(p) =: \Sigma_A(p^2)\,\not{p} - \Sigma_M(p^2)\,. \tag{11.9}$$

An analogous decomposition holds for the inverse propagator itself:

$$iS^{-1}(p) = A(p^2)\,(\not{p} - M(p^2))\,, \tag{11.10}$$

which defines the **fermion mass function** $M(p^2)$, and $1/A(p^2)$ is called the fermion 'wave-function renormalization'. When substituting both equations into Eq. (11.8) we find the perturbative expansion of these dressing functions:

$$A(p^2) = Z_\psi + \Sigma_A(p^2)\,, \qquad A(p^2)M(p^2) = Z_\psi Z_m m + \Sigma_M(p^2)\,. \tag{11.11}$$

Likewise, the only two possible tensors for the photon vacuum polarization are $g^{\mu\nu}$ and $q^\mu q^\nu$, so we can write

$$\Pi^{\mu\nu}(q) = a(q^2)\, g^{\mu\nu} + b(q^2)\, q^\mu q^\nu \,. \tag{11.12}$$

The scalar functions a and b cannot have poles at $q^2 = 0$ because that would correspond to an intermediate massless particle; but since the vacuum polarization is already the sum of all 1PI diagrams, intermediate propagators are excluded by definition. Now, the Ward identity $q_\mu \Pi^{\mu\nu} = 0$ entails that the vacuum polarization must be transverse to the photon momentum and therefore $a = -b\, q^2$. The only remaining tensor structure is then

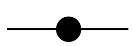
$$\Pi^{\mu\nu}(q) = \Pi(q^2)\,(q^2\, g^{\mu\nu} - q^\mu q^\nu) = q^2\, \Pi(q^2)\, T_q^{\mu\nu} \tag{11.13}$$

which is proportional to the transverse projector, however with an additional factor q^2 in front.[9] From the geometric resummation of the photon propagator analogous to Eq. (11.7) it is immediately clear that all longitudinal parts will be annihilated by $\Pi^{\mu\nu}(q)$, except for the leading tree-level term that contains the gauge parameter λ. Therefore, the longitudinal part of the photon propagator does not pick up any loop corrections beyond tree-level:

$$i(D^{-1})^{\mu\nu}(q) = -q^2 \left(\frac{T_q^{\mu\nu}}{D(q^2)} + \lambda\, L_q^{\mu\nu} \right), \qquad D^{-1}(q^2) = Z_A - \Pi(q^2)\,. \tag{11.14}$$

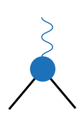
Because the longitudinal part does not get dressed, it contains no divergences and does not need to be renormalized either. This is precisely the origin of the second constraint in Eq. (11.3). As another consequence, the global factor q^2 in front of the bracket remains and, after inversion, becomes a factor $1/q^2$ in the photon propagator. Hence the photon remains massless, even with interactions, due to gauge invariance!

After inverting the above formulas, the general expressions for the fully dressed propagators and the dressed vertex become



$$S(p) = \frac{i}{A(p^2)}\, \frac{\not p + M(p^2)}{p^2 - M^2(p^2) + i\epsilon}\,, \tag{11.15}$$

$$D^{\mu\nu}(q) = -\frac{i}{q^2 + i\epsilon} \left(D(q^2)\, T_q^{\mu\nu} + \frac{1}{\lambda}\, L_q^{\mu\nu} \right), \tag{11.16}$$



$$ig\, \Gamma^\mu(p,q) = ig\big(f_1(p^2, q^2, p\cdot q)\, \gamma^\mu + \dots \big)\,. \tag{11.17}$$

The fermion-photon vertex is more complicated because it depends on two momenta, which leads to 12 possible tensors (we will return to this point later). In any case, when we write the vertex correction as $\Omega^\mu(p,q) = V_1(p^2, q^2, p\cdot q)\, \gamma^\mu + \dots$, where the dots refer to the remaining tensor structures, the general form of the vertex dressing of γ^μ is:

$$f_1(p^2, q^2, p\cdot q) = Z_\psi + V_1(p^2, q^2, p\cdot q)\,. \tag{11.18}$$

[9]Had we solved for $b = -a/q^2$ instead, b would pick up a pole at $q^2 = 0$ contrary to what we just observed.

Renormalization conditions. The next step is to impose the renormalization conditions that are necessary to eliminate the three renormalization constants. We demand that the fermion and photon propagators become free propagators at the respective pole location, which entails

$$A(p^2 = m^2) \stackrel{!}{=} 1\,, \qquad M(p^2 = m^2) \stackrel{!}{=} m\,, \qquad D(q^2 = 0) \stackrel{!}{=} 1\,. \tag{11.19}$$

This determines the renormalization constants via Eqs. (11.11) and (11.14):

$$Z_\psi = 1 - \Sigma_A(m^2)\,, \qquad Z_\psi m_B = m - \Sigma_M(m^2)\,, \qquad Z_A = 1 + \Pi(0)\,. \tag{11.20}$$

The resulting dressing functions, which are now finite, become

$$\begin{aligned} A(p^2) &= 1 + \Sigma_A(p^2) - \Sigma_A(m^2)\,, \\ A(p^2)M(p^2) &= m + \Sigma_M(p^2) - \Sigma_M(m^2)\,, \\ D^{-1}(q^2) &= 1 - \Pi(q^2) + \Pi(0)\,. \end{aligned} \tag{11.21}$$

We could impose another condition on the vertex,

$$f_1(m^2,0,0) \stackrel{!}{=} 1 \quad \Rightarrow \quad Z_\psi = 1 - V_1(m^2,0,0)\,, \tag{11.22}$$

but this is not necessary because it is already guaranteed by the Ward identity which allowed us to relate Z_g with Z_A. We will later see that $V_1(m^2,0,0) = \Sigma_A(m^2)$ is automatically satisfied in the one-loop calculation. More generally, we will also see this directly from the nonperturbative form of the vertex that follows from gauge invariance.

In summary we arrive at analogous conclusions as for the scalar theory: we can eliminate the UV divergences from the theory by imposing three renormalization conditions. We chose an onshell renormalization to make a direct connection with experiment, but our choice of renormalization conditions is arbitrary. In turn, the renormalized mass m and the renormalized charge $g = e$ are no longer predictions of the theory but they must be taken from experiment.

Fermion self-energy. As a concrete example, let us work out the one-loop contribution to the fermion self-energy in Fig. 11.1. It has the form

$$i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4}\,(ig\gamma_\mu)\,S_0(k)\,(ig\gamma_\nu)\,D_0^{\mu\nu}(p-k)\,. \tag{11.23}$$

We can ignore all renormalization constants that enter through the Feynman rules in Eq. (11.5) because they do not contribute at one-loop; the same is true for the mass renormalization so we can simply set $m_B = m$. In Feynman gauge the photon propagator is proportional to $g^{\mu\nu}$ and therefore the integral becomes

$$i\Sigma(p) = -g^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu(\not{k} + m)\gamma_\mu}{\left[k^2 - m^2 + i\epsilon\right]\left[(p-k)^2 + i\epsilon\right]}\,. \tag{11.24}$$

Here we can exploit the formula (8.13) that we derived in the scalar theory after employing Feynman parameters and performing a Wick rotation:

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{\prod_{i=1}^2 \left[(k+p_i)^2 - m_i^2 + i\epsilon\right]} = i\int_0^1 dx\, I_2^{(d)}\,, \tag{11.25}$$

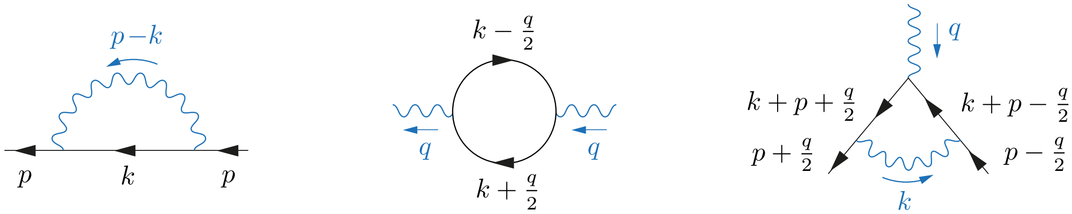


FIGURE 11.1: One-loop contributions to the fermion self-energy, the vacuum polarization and the vertex correction.

with $I_2^{(d)}$ defined in Eq. (8.20) and the remaining quantities in Eqs. (8.6–8.7). Since we want to carry on with dimensional regularization we already wrote the formula in d spacetime dimensions. In our present example we have $p_1 = -p$ and $p_2 = 0$, $m_1 = 0$ and $m_2 = m$, and $x_1 = x$, $x_2 = 1-x$ and therefore

$$\Delta = (1-x)(m^2 - xp^2), \qquad k^\mu = l^\mu + xp^\mu \,. \tag{11.26}$$

Thus, the self-energy becomes

$$i\Sigma(p) = -ig^2 \int_0^1 dx \int \frac{d^d l_E}{(2\pi)^d} \, \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(l_E^2+\Delta)^2}\Big|_{k\to l+xp} \,. \tag{11.27}$$

We still have to work on the numerator. In d dimensions $\delta^\mu{}_\mu = d$, and the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ entails $\gamma^\mu\gamma_\mu = \delta^\mu{}_\mu = d$. This leads to $\gamma^\mu \not{k}\gamma_\mu = (2-d)\not{k}$ and finally

$$\gamma^\mu(\not{k}+m)\gamma_\mu = (2-d)\not{k} + md = (2-d)(\not{l} + x\not{p}) + md \,. \tag{11.28}$$

Factors of l^μ in the numerator are easily manageable because

$$\begin{aligned} &\int \frac{d^d l_E}{(2\pi)^d} \, \frac{l^\mu}{(l_E^2+\Delta)^2} = 0 \,, \\ &\int \frac{d^d l_E}{(2\pi)^d} \, \frac{l^\mu l^\nu}{(l_E^2+\Delta)^2} = -\frac{1}{d}\, g^{\mu\nu} \int \frac{d^d l_E}{(2\pi)^d} \, \frac{l_E^2}{(l_E^2+\Delta)^2} \,. \end{aligned} \tag{11.29}$$

The first integral vanishes due to symmetry (replace $l \to -l$ in the integrand), and so does the second for $\mu \neq \nu$. For $\mu = \nu$ it must be proportional to $g^{\mu\nu}$ by Lorentz invariance, and by contracting the indices one verifies that the prefactor on the r.h.s. is correct. Hence, Eq. (11.28) becomes

$$\gamma^\mu(\not{k}+m)\gamma_\mu = (2-d)\, x\not{p} + md \quad \Rightarrow \quad i\Sigma(p) = -ig^2 \int_0^1 dx \left[(2-d)\, x\not{p} + md\right] I_2^{(d)} .$$

By comparing with Eq. (11.9) we read off the self-energy contributions:

$$\Sigma_A(p^2) = g^2(d-2)\int dx\, x\, I_2^{(d)}, \qquad \Sigma_M(p^2) = g^2 md \int dx\, I_2^{(d)} \,. \tag{11.30}$$

Setting now $d = 4 - \epsilon$, taking the limit $\epsilon \to 0$, and inserting the result (8.29) for $I_2^{(d)}$ in dimensional regularization, we arrive at

$$\begin{aligned}
\Sigma_A(p^2) &= \frac{\alpha}{2\pi} \int dx\, x \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{\Delta} - 1 \right], \\
\Sigma_M(p^2) &= \frac{\alpha m}{\pi} \int dx \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{\Delta} - \frac{1}{2} \right],
\end{aligned} \tag{11.31}$$

where we have also replaced the coupling by $g^2 = 4\pi\alpha$. This is the one-loop fermion self-energy in dimensional regularization. We see that the method is completely analogous to the scalar theory; from Eq. (11.30) we could have equally derived the result in Pauli-Villars regularization. In both cases the expressions contain divergent and finite pieces: in dimensional regularization the divergences are of the form $\sim 1/\epsilon$ whereas with PV regularization they are logarithmic.

To arrive at finite expressions, we apply the renormalization procedure outlined above. That is, we subtract the self-energy at $p^2 = m^2$:

$$\begin{aligned}
\Sigma_A(p^2) - \Sigma_A(m^2) &= \frac{\alpha}{2\pi} \int dx\, x \ln \frac{\Delta_m}{\Delta}, \\
\Sigma_M(p^2) - \Sigma_M(m^2) &= \frac{\alpha m}{\pi} \int dx \ln \frac{\Delta_m}{\Delta},
\end{aligned} \qquad \frac{\Delta_m}{\Delta} = \frac{m^2(1-x)}{m^2 - x p^2}, \tag{11.32}$$

which makes the dressing functions in Eq. (11.21) finite. Note that the logarithm develops a branch cut for negative arguments. Since $0 < x < 1$, the condition is

$$\frac{p^2}{m^2} > \frac{1}{x} > 1, \tag{11.33}$$

and therefore the branch cut starts at $p^2 = m^2$. This is just what we anticipated with the Källén-Lehmann representation, cf. Fig. 6.2. Due to the self-energy correction the fermion can split into a fermion plus a photon (and, when going to higher orders in perturbation theory, arbitrarily many photons), but since the photon is massless, the multiparticle continuum that produces the cut starts at $p^2 = (m + m_\gamma)^2 = m^2$.

From Eq. (11.21) we extract the one-loop result for the mass function $M(p^2)$:

$$\begin{aligned}
M(p^2) &= \frac{m + \Sigma_M(p^2) - \Sigma_M(m^2)}{1 + \Sigma_A(p^2) - \Sigma_A(m^2)} \\
&\approx m + \Sigma_M(p^2) - \Sigma_M(m^2) - m\left(\Sigma_A(p^2) - \Sigma_A(m^2)\right) \\
&= m\left[1 + \frac{\alpha}{\pi} \int dx \left(1 - \frac{x}{2}\right) \ln \frac{\Delta_m}{\Delta}\right],
\end{aligned} \tag{11.34}$$

which inherits the branch cut for $p^2 > m^2$. It is also instructive to work out the explicit form for large spacelike $Q^2 := -p^2 \gg m^2$. In that case

$$\ln \frac{\Delta_m}{\Delta} \approx \ln \frac{m^2(1-x)}{x\, Q^2} \approx -\ln \frac{Q^2}{m^2} + \dots \tag{11.35}$$

and therefore the mass function falls off logarithmically with Q^2 (see Fig. 11.2):

$$M(p^2) = m\left[1 - \frac{\alpha}{\pi} \ln \frac{Q^2}{m^2} \int dx \left(1 - \frac{x}{2}\right) + \dots\right] = m\left[1 - \frac{3\alpha}{4\pi} \ln \frac{Q^2}{m^2} + \dots\right]. \tag{11.36}$$

Vacuum polarization. We already discussed the general properties of the vacuum polarization (transversality and analyticity at $q^2 = 0$) above. The generally allowed form for the vacuum polarization tensor is

$$\Pi^{\mu\nu}(q) = \Pi(q^2)\,(q^2 g^{\mu\nu} - q^\mu q^\nu) + \widetilde{\Pi}(q^2)\,g^{\mu\nu}\,, \tag{11.37}$$

but the Ward identity $q_\mu \Pi^{\mu\nu} = 0$ enforces $\widetilde{\Pi}(q^2) = 0$. This is indeed true at each order in perturbation theory provided that the regularization method respects gauge invariance.

One can treat the one-loop expression in Fig. 11.1 in complete analogy to the fermion self-energy example. The one-loop Feynman graph has the form

$$\begin{aligned} i\Pi^{\mu\nu}(q) &= -\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\,(ig\gamma^\mu)\,S_0(k_+)\,(ig\gamma^\nu)\,S_0(k_-) \\ &= -g^2\,\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu(\not{k}_+ + m)\gamma^\nu(\not{k}_- + m)}{\big[k_+^2 - m^2 + i\epsilon\big]\big[k_-^2 - m^2 + i\epsilon\big]}\,. \end{aligned} \tag{11.38}$$

The calculation is a bit lengthier but still manageable; the result is **(Ex)**

$$\begin{aligned} \Pi(q^2) &= -8g^2\int dx\,x(1-x)\,I_2^{(d)}\,, \\ \widetilde{\Pi}(q^2) &= -4g^2\int dx\,\Big(I_2^{(d)}\Delta + (1-\tfrac{2}{d})\,\widetilde{I}_2^{(d)}\Big) \end{aligned} \tag{11.39}$$

with $\Delta = m^2 - x(1-x)\,q^2$. The integrals are given in Eq. (8.32), and with their explicit form it is easy to check that $\widetilde{\Pi}(q^2)$ vanishes indeed in dimensional regularization. However, this is not true for a momentum cutoff: in that case $\widetilde{\Pi}(q^2)$ is not only nonzero but also develops a quadratic divergence (as one would infer from a dimensional analysis of the diagram) due to the appearance of $\widetilde{I}_2^{(d)}$. Hence, a cutoff regulator breaks gauge invariance, and therefore it is not the optimal choice when dealing with gauge theories (unless one knows how to eliminate the contamination from such 'gauge parts').

The transverse piece, on the other hand, is only logarithmically divergent. In dimensional regularization it is given by

$$\Pi(q^2) = -\frac{2\alpha}{\pi}\int dx\,x(1-x)\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi M^2}{\Delta}\right). \tag{11.40}$$

After performing the subtraction in Eq. (11.21) it becomes

$$\Pi(q^2) - \Pi(0) = -\frac{2\alpha}{\pi}\int dx\,x(1-x)\,\ln\frac{\Delta_0}{\Delta}\,, \qquad \frac{\Delta_0}{\Delta} = \frac{m^2}{m^2 - x(1-x)\,q^2}\,. \tag{11.41}$$

Notice again the branch cut from the logarithm: since $0 < x(1-x) < \frac{1}{4}$ the condition is now

$$\frac{q^2}{m^2} > \frac{1}{x(1-x)} > 4 \qquad \Rightarrow \qquad q^2 > 4m^2 \tag{11.42}$$

as it should be, because $2m$ is the threshold for e^+e^- pair creation.

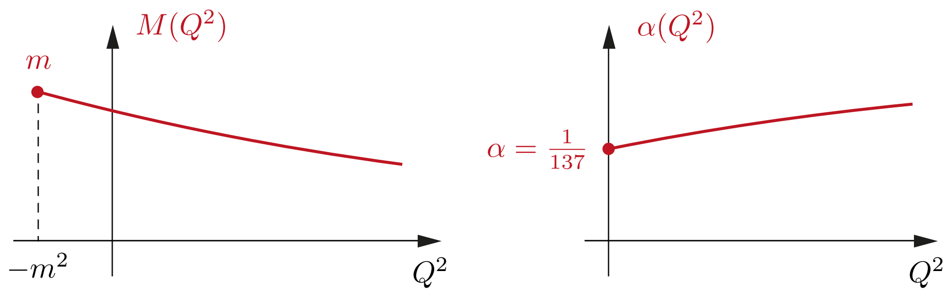


FIGURE 11.2: One-loop behavior of the fermion mass function and running coupling.

Running coupling. The vacuum polarization has another practical relevance. We can define an effective **running coupling** as the product of the coupling constant α and the photon dressing:

$$\alpha(q^2) := \alpha\, D(q^2) = \frac{\alpha}{1 - \Pi(q^2) + \Pi(0)} \,. \tag{11.43}$$

It is fully determined by the vacuum polarization, so for positive $q^2 > 4m^2$ it inherits the branch cut from Eq. (11.41). To obtain the form for large spacelike $Q^2 := -q^2 \gg m^2$ we plug in the one-loop result:

$$\frac{\Delta_0}{\Delta} \approx \frac{m^2}{x(1-x)\,Q^2} \quad \Rightarrow \quad \Pi(q^2) - \Pi(0) = \frac{2\alpha}{\pi}\left[\ln\frac{Q^2}{m^2}\underbrace{\int dx\, x(1-x)}_{1/6} + \dots\right],$$

and we find that the running coupling rises logarithmically with Q^2 as in Fig. 11.2:

$$\alpha(q^2) \approx \frac{\alpha}{1 - \frac{\alpha}{3\pi}\,\ln\frac{Q^2}{m^2}} \,. \tag{11.44}$$

The reason behind the definition (11.43) is the following. Suppose we reformulate our renormalization conditions (11.19) in terms of $A(p^2)$, $M(p^2)$ and $\alpha(q^2)$:

$$A(p^2 = m^2) \stackrel{!}{=} 1\,, \qquad M(p^2 = m^2) \stackrel{!}{=} m\,, \qquad \alpha(q^2 = 0) \stackrel{!}{=} \alpha\,, \tag{11.45}$$

which makes the nature of m and α as an external input to QED explicit. With this we can calculate the momentum dependence of $M(p^2)$ and $\alpha(q^2)$, for example in one-loop perturbation theory as in Fig. 11.2. However, these curves would look the *same* if we had not renormalized at $p^2 = m^2$ and $q^2 = 0$ but at some arbitrary scales $p^2 = \mu^2$ and $q^2 = \nu^2$, provided that we used $M(\mu^2)$ and $\alpha(\nu^2)$ as the new input values. Such a change cannot affect $M(p^2)$, $\alpha(q^2)$ nor any other prediction of the theory. On the other hand, this only holds as long as $M(p^2)$ and $\alpha(q^2)$ do not *additionally* depend on the renormalization point (that is, they must be *renormalization-group invariant*).

This is true for the fermion mass function, which we can easily confirm. From the relations (11.2) between the bare and renormalized fields one immediately derives the relations between the bare and renormalized n-point functions, for example

$$\langle\Omega|\,\mathsf{T}\,\psi(x)_\mathrm{B}\,\bar\psi_\mathrm{B}(y)\,|\Omega\rangle = Z_\psi\,\langle\Omega|\,\mathsf{T}\,\psi(x)\,\bar\psi(y)\,|\Omega\rangle \tag{11.46}$$

and therefore $S_\mathrm{B}(p) = Z_\psi\, S(p)$. When we impose the renormalization conditions at some renormalization point $p^2 = \mu^2$, Z_ψ will depend on the renormalization point and so will all the Green functions of the theory. For example, the renormalized propagator is $S(p,\mu)$ and its dressing functions $A(p^2)$ and $M(p^2)$ have the form

$$A_{\mathrm{B}}(p^2) = \frac{1}{Z_\psi(\mu^2)}\, A(p^2,\mu^2) \qquad \text{but} \qquad M_{\mathrm{B}}(p^2) = M(p^2)\,. \tag{11.47}$$

Due to our definition (11.10) the dependence on the renormalization constant Z_ψ is entirely carried by the function $A(p^2)$, whereas $M(p^2)$ stays unrenormalized: $M_B(p^2) = M(p^2)$. The divergences must therefore cancel in the mass function even if we had not renormalized the theory. Because there is no Z factor that relates the bare with the renormalized mass function, $M(p^2)$ also cannot depend on μ and its interpretation as a 'running fermion mass' is acceptable.

The analogous combination for the coupling α must be of the form

$$\big(g^2 f(q^2)\big)_{\mathrm{B}} = g^2 f(q^2)\,. \tag{11.48}$$

The relation $Z_g\, Z_A^{1/2}$ in Eq. (11.3) that follows from the Ward identity suggests to identify $f(q^2)$ with the photon dressing function, because also

$$\big(g^2 D(q^2)\big)_{\mathrm{B}} = Z_g^2\, Z_A\, \big(g^2 D(q^2)\big) = g^2 D(q^2) \tag{11.49}$$

stays unrenormalized, i.e., it is a renormalization-group invariant.

Since the values for m and α are an input to QED, the theory 'knows' how the electron mass and its charge evolve with the momentum scale, and this scale dependence is encoded in the functional form of $M(p^2)$ and $\alpha(q^2)$. With $e^2 = 4\pi\alpha$ we may interpret $\alpha(q^2)$ as the effective momentum dependence of the electron charge. Nonrelativistically, the spacelike Q^2 dependence translates into a potential between two electrons (or an electron and a positron). If we pull two electrons infinitely far apart, we probe the coupling at $Q^2 = 0$; this is where we extract $\alpha(0) \approx \frac{1}{137}$ experimentally. The rise of $\alpha(q^2)$ at $Q^2 > 0$ can be viewed as a **screening of the charge**: at $Q^2 = 0$, the electron is screened by a cloud of virtual e^+e^- pairs, but at higher Q^2 (smaller distances) we eventually penetrate this charge cloud and see more of the electron's 'true' charge which is larger. Hence the name 'vacuum polarization', because the vacuum behaves like a polarizable medium.

On the other hand, the rise of $\alpha(q^2)$ happens extremely slowly and the coupling remains the same over many orders of magnitude. Between $Q^2 = 0 \dots 30\,\mathrm{GeV}^2$, this rise is only about 1% from the e^+e^- loop and $\sim 5\%$ in total (including heavier leptons and also quarks). This is good news because $\alpha(q^2)$ is also the expansion parameter in perturbation theory. The result in Fig. 11.2 was obtained at one-loop; if $\alpha(q^2)$ would rise dramatically with the momentum, we could forget about applying perturbation theory at larger Q^2. Fortunately the coupling is still small at large momenta, so the perturbative treatment is justified.

Nevertheless, the fact that this rise continues indefinitely casts doubt on the behavior of the theory at very small distances or very large energies; it is referred to as the **Landau pole** of QED. The one-loop formula develops a pole at $Q^2 \sim (10^{277}\,\mathrm{GeV})^2$, which is completely irrelevant in practice because electromagnetism eventually merges with the weak interactions and even the Planck scale 10^{19} GeV is much lower. Still, this implies that QED by itself is not a well-defined theory at high energies. The situation in QCD is reversed: $\alpha(Q^2)$ falls off with higher momenta due to asymptotic freedom, so the theory is well-defined in the ultraviolet.

Fermion-photon vertex. Before discussing the perturbative one-loop result for the vertex correction in Fig. 11.1, let's have a look at the general properties of the fermion-photon vertex. We use the same kinematics as earlier: $q = p_+ - p_-$ is the incoming photon momentum and $p = (p_+ + p_-)/2$ is the average momentum of the fermions. The squared fermion momenta are given by

$$p_\pm^2 = p^2 + \frac{q^2}{4} \pm p\cdot q \qquad \Rightarrow \qquad p_+^2 - p_-^2 = 2\,p\cdot q\,, \tag{11.50}$$

so the onshell limit $p_\pm^2 = m^2$ corresponds to $p\cdot q = 0$ and $p^2 = m^2 - q^2/4$. The dependence of the vertex on two independent momenta leads to 12 possible tensors:

$$\{\gamma^\mu,\ p^\mu,\ q^\mu\} \times \big\{\mathbb{1},\ \not{p},\ \not{q},\ [\not{p},\not{q}]\big\}\,. \tag{11.51}$$

This looks rather hopeless, but fortunately gauge invariance provides us with some ordering principle. We mentioned that local $U(1)$ gauge invariance is equivalent to the statement that the photons couple to fermions through the conserved vector current of the global $U(1)$ symmetry. A current that is classically conserved induces **Ward-Takahashi identities (WTIs)** for the Green functions of the theory. These are identities that relate an $n-$point function to $(n-1)-$point functions. Without proof, we state the WTI for the fermion-photon vertex:

$$q_\mu\,\Gamma^\mu(p,q) = iS^{-1}(p+\tfrac{q}{2}) - iS^{-1}(p-\tfrac{q}{2})\,, \tag{11.52}$$

which holds not only for $q^\mu \to 0$ but in general, and it tells us that the vertex is partially determined by the inverse fermion propagator.

The WTI has several practical consequences. First, we can work it out explicitly using the tensor decomposition (11.10) for the inverse fermion propagator. Abbreviating $B(p^2) = A(p^2)M(p^2)$, as well as $A(p_\pm^2) = A_\pm$ and $B(p_\pm^2) = B_\pm$, it takes the form

$$\begin{aligned} q_\mu\,\Gamma^\mu(p,q) &= \left(\not{p} + \frac{\not{q}}{2}\right)A_+ - \left(\not{p} - \frac{\not{q}}{2}\right)A_- - B_+ + B_- \\ &= \underbrace{\frac{A_+ + A_-}{2}}_{=:\,\overline{A}}\,\not{q} + \underbrace{\frac{A_+ - A_-}{2\,p\cdot q}}_{=:\,\Delta_A}\,2\,p\cdot q\,\not{p} - \underbrace{\frac{B_+ - B_-}{2\,p\cdot q}}_{=:\,\Delta_B}\,2\,p\cdot q \\ &= q_\mu\big[\underbrace{\overline{A}\,\gamma^\mu + 2p^\mu\,(\Delta_A\,\not{p} - \Delta_B)}_{=:\,\Gamma^\mu_{\text{BC}}(p,q)}\big]. \end{aligned} \tag{11.53}$$

The quantities Δ_A and Δ_B are difference quotients because $2\,p\cdot q = p_+^2 - p_-^2$, and in the limit $p_+^2 = p_-^2$ they become the derivatives of the dressing functions $A(p^2)$ and $B(p^2)$ with respect to p^2. The bracket in the last line defines the **Ball-Chiu vertex** which is the part of the vertex that is constrained by gauge invariance. Consequently, the full vertex can only differ by a purely transverse part that does not contribute to the WTI:

$$\Gamma^\mu(p,q) = \Gamma^\mu_{\text{BC}}(p,q) + \Gamma^\mu_{\text{T}}(p,q)\,. \tag{11.54}$$

The transverse part cannot have analytic poles at $q^2 = 0$ because that would again contradict its 1PI property. In combination with the transversality condition, one can show that this has the consequence that the transverse part must be at least linear in q^μ, so it vanishes for $q^\mu \to 0$. It depends on eight remaining tensors, and in analogy to the vacuum polarization one can construct appropriate tensor bases so that its dressing functions are free of kinematic singularities and constraints at $q^2 = 0$.

We are now ready to verify the first relation in Eq. (11.3). It follows from the fact that the WTI holds for renormalized and unrenormalized quantities alike: if we define $Z_\Gamma = Z_\psi \, Z_g \, Z_A^{1/2}$, then the argument that we used in Eq. (11.46) entails

$$\Gamma_{\mathrm{B}} = \frac{1}{Z_\Gamma}\,\Gamma\,, \qquad S_{\mathrm{B}}^{-1} = \frac{1}{Z_\psi}\,S^{-1}\,. \tag{11.55}$$

Therefore, the WTI for the bare vertex and propagator is identical to the renormalized WTI in Eq. (11.52), except that an additional factor $1/Z_\Gamma$ appears on the left-hand side and $1/Z_\psi$ on the right. This, in turn, requires $Z_\Gamma = Z_\psi$ and consequently $Z_g \, Z_A^{1/2} = 1$.

Since that identity eliminates the renormalization constant Z_g we had no freedom anymore to renormalize the vertex. Instead, we claimed that $f(m^2, 0, 0) = 1$ will be automatically ensured by the WTI. Now we can see how this comes about: we renormalized the fermion propagator at $p^2 = m^2$ and the photon propagator at $q^2 = 0$; the corresponding onshell limit for the vertex is

$$\Gamma^\mu(p,q) \to A(m^2)\,\gamma^\mu + 2p^\mu\left(A'(m^2)\,\not{p} - B'(m^2)\right). \tag{11.56}$$

This is the *exact* form of the vertex in that limit because the transverse part does not contribute. With our renormalization condition $A(m^2) = 1$ the dressing function of the the γ^μ component is indeed $f(m^2, 0, 0) = A(m^2) = 1$, as advertised.

Electromagnetic form factors. In onshell scattering matrix elements we additionally need to attach Dirac spinors to the vertex, so we must work out the quantity $\bar{u}(p_+)\,\Gamma^\mu(p,q)\,u(p_-)$. In onshell kinematics $p\cdot q = 0$ and $p^2 = m^2 - q^2/4$, so q^2 is the only remaining Lorentz-invariant. In that case the WTI (11.52) reduces to the **Ward identity**

$$q_\mu\,\bar{u}(p_+)\,\Gamma^\mu(p,q)\,u(p_-) = 0\,. \tag{11.57}$$

It follows immediately from taking (11.56) in the onshell limit $p\cdot q = 0$ and exploiting the Dirac equation for the onshell spinors:

$$\begin{array}{l} \bar{u}(p_+)\,\not{p}_+ = m\,\bar{u}(p_+) \\ \not{p}_-\,u(p)_- = m\,u(p_-) \end{array} \quad \Rightarrow \quad \bar{u}(p_+)\,\not{q}\,u(p_-) = \bar{u}(p_+)\,(\not{p}_+ - \not{p}_-)\,u(p_-) = 0\,. \tag{11.58}$$

On the other hand, starting from the tensor structures in Eq. (11.51) we can write down the most general onshell decomposition of the current. By judicious use of the Dirac equations one can eliminate all slashed quantities, for example

$$\bar{u}(p_+)\,\not{p}\,u(p_-) = \bar{u}(p_+)\,\frac{\not{p}_+ + \not{p}_-}{2}\,u(p_-) = mu(p_-)\,. \tag{11.59}$$

This leaves three possible dressing functions which can only depend on q^2:

$$\bar{u}(p_+)\,\Gamma^\mu(p,q)\,u(p_-) = \bar{u}(p_+)\left[a\,\gamma^\mu + b\,p^\mu + c\,q^\mu\right]u(p_-)\,. \tag{11.60}$$

The Ward identity then enforces $c=0$, so we are left with γ^μ and p^μ. Using the Dirac equations it is easy to prove the **Gordon identity** **(Ex)**

$$\bar{u}(p_+)\left[\gamma^\mu - \frac{p^\mu}{m} - \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p_-)\,, \tag{11.61}$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$, which allows us to eliminate p^μ in favor of $\sigma^{\mu\nu}q_\nu$. The final result for the onshell vertex is

$$\bar{u}(p_+)\,\Gamma^\mu(p,q)\,u(p_-) = \bar{u}(p_+)\left[F_1(q^2)\,\gamma^\mu + F_2(q^2)\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p_-)\,. \tag{11.62}$$

$F_1(q^2)$ and $F_2(q^2)$ are called the **electromagnetic Dirac and Pauli form factors**, respectively, and $F_2(0)$ is the anomalous magnetic moment of the fermion.

Along the same lines we can also apply spinors to Eq. (11.56), which amounts to replacing $\not{p}\to m$ and eliminating p^μ using the Gordon identity. The form factors become

$$F_1(q^2) = A(m^2) - C(m^2) + q^2\,[\dots]\,, \qquad F_2(q^2) = C(m^2) + [\dots]\,, \tag{11.63}$$

where $C(m^2) := -2m\left(mA'(m^2) - B'(m^2)\right) = 2m\,A(m^2)\,M'(m^2)$ and the dots refer to further contributions coming from the transverse part of the vertex. Observe that the renormalization condition $A(m^2)=1$ does *not* lead to $F_1(0)=1$; we have to choose $A(m^2) = 1 + C(m^2)$ instead. This is equivalent to the following modification of the renormalization conditions in Eq. (11.19):

$$iS^{-1}(\not{p})\Big|_{\not{p}\to m} \stackrel{!}{=} 0 \;\;\Rightarrow\;\; M(m^2) = m\,, \qquad \frac{d}{d\not{p}}\,iS^{-1}(\not{p})\Big|_{\not{p}\to m} \stackrel{!}{=} 1 \;\;\Rightarrow\;\; A(m^2) = 1 + C(m^2)\,. \tag{11.64}$$

Here we view the propagator as a function of $\not{p}$, with $\not{p}^2 = p^2$, and thereby also take the derivative of the dressing functions. This will also modify the renormalization constants in Eq. (11.20),

$$Z_\psi = 1 + C(m^2) - \Sigma_A(m^2)\,, \qquad Z_\psi m_B = \left(1 + C(m^2)\right)m - \Sigma_M(m^2)\,, \tag{11.65}$$

as well as the result for the renormalized dressing functions:

$$\begin{aligned} A(p^2) &= 1 + C(m^2) + \Sigma_A(p^2) - \Sigma_A(m^2)\,,\\ A(p^2)M(p^2) &= \left(1 + C(m^2)\right)m + \Sigma_M(p^2) - \Sigma_M(m^2)\,. \end{aligned} \tag{11.66}$$

If we write $F_1(q^2) = Z_\psi + \delta F_1(q^2)$, then the Ward identity gives the result $\delta F_1(0) = \Sigma_A(m^2) - C(m^2)$.

Perturbative result for the vertex correction. The one-loop calculation for the vertex correction in Fig. 11.1 is considerably more complicated than the self-energy calculation but otherwise completely analogous. Starting from Eq. (11.8), the diagram is given by

$$\begin{aligned} ig\,\Omega^\mu(p,q) &= \bar{u}(p_+)\int\frac{d^4k}{(2\pi)^4}\,(ig\,\gamma_\rho)\,S(k_+)\,(ig\,\gamma_\mu)\,S(k_-)\,(ig\,\gamma_\sigma)\,D^{\rho\sigma}(k)\,u(p_-)\\ &= g^3\int\frac{d^4k}{(2\pi)^4}\,\frac{\bar{u}(p_+)\,\gamma_\rho\,(\not{k}_+ + m)\,\gamma_\mu\,(\not{k}_- + m)\,\gamma^\rho\,u(p_-)}{\left[k_+^2 - m^2 + i\epsilon\right]\left[k_-^2 - m^2 + i\epsilon\right]\left[k^2 + i\epsilon\right]}\,. \end{aligned} \tag{11.67}$$

Inserting the formula (8.13) with $p_1 = p + \frac{q}{2}$, $p_2 = p - \frac{q}{2}$, $p_3 = 0$, $m_1 = m_2 = m$ and $m_3 = 0$ yields

$$\Omega^\mu(p,q) = -2g^2\int\underbrace{dx\,dy\,dz\,\delta(x+y+z-1)}_{:=\,d\omega}\int\frac{d^4l_E}{(2\pi)^4}\,\frac{\mathcal{N}}{(l_E^2+\Delta)^3}\,, \tag{11.68}$$

where $\mathcal{N}$ is the numerator with the replacement $k^\mu \to l^\mu - a\,p^\mu - \frac{b}{2}\,q^\mu$ and the function in the denominator is given by

$$\Delta = a^2 m^2 + (b^2 - a^2)\,\frac{q^2}{4}\,, \qquad a = x + y = 1 - z\,, \quad b = x - y\,. \tag{11.69}$$

The hardest part is working out the numerator. After some pages of calculation, the vertex correction becomes

(Ex)

$$\Omega^\mu(p,q) = 2g^2 \int d\omega\;\bar{u}(p_+)\left[H_1\,\gamma^\mu + H_2\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p_-)\,, \tag{11.70}$$

where

$$\begin{aligned} H_1 &= \frac{(d-2)^2}{d}\,\widetilde{I}_3^{(d)} + 2\left(2m^2\left(a\,(1+a) - 1\right) - \Delta + (1-a)\,q^2\right) I_3^{(4)}\,, \\ H_2 &= 4m^2 a\,(1-a)\,I_3^{(4)}\,. \end{aligned} \tag{11.71}$$

The divergent parts can only come from factors $\sim l_E^2$ in the numerator which lead to the divergent integral $\widetilde{I}_3^{(d)}$. All other contributions are finite and proportional to $I_3^{(4)}$, so we already took the limit $d \to 4$ for those. Note in particular that H_2 is finite, i.e., the Pauli form factor is free of divergences.

Using dimensional regularization for $\widetilde{I}_3^{(d)}$, the form factors become

$$\begin{aligned} F_1(q^2) &= Z_\psi + \frac{\alpha}{2\pi}\int d\omega\left[\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi M^2}{\Delta} - 3 + \frac{2m^2\left(a(1+a)-1\right) + (1-a)\,q^2}{\Delta}\right], \\ F_2(q^2) &= m^2\,\frac{\alpha}{\pi}\int d\omega\;\frac{a(1-a)}{\Delta}\,. \end{aligned} \tag{11.72}$$

At $q^2 = 0$, we have $\Delta = a^2 m^2$, and with $a = 1 - z$ from Eq. (11.69) we find

$$F_2(0) = \frac{\alpha}{\pi}\int d\omega\;\frac{z}{1-z} = \frac{\alpha}{\pi}\int_0^1 dz \int_0^{1-z} dy\;\frac{z}{1-z} = \frac{\alpha}{\pi}\int_0^1 dz\,z = \frac{\alpha}{2\pi}\,. \tag{11.73}$$

This is Schwinger's famous result for the **anomalous magnetic moment of the electron** at one-loop order. Inserting $\alpha \approx \frac{1}{137}$, the numerical value is about 1‰: $F_2(0) = 0.0011614$, plus higher orders in perturbation theory. Compare this with the experimental result: $F_2(0)_{\text{exp}} = 0.0011597$.

Another check is whether the Ward identity truly holds. From Eq. (11.72) we infer

$$F_1(0) = Z_\psi + \frac{\alpha}{2\pi}\left[\frac{1}{2}\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi M^2}{m^2} - 1\right) - \int da\left(\frac{2(1-a)}{a} + a\,\ln a^2\right)\right]. \tag{11.74}$$

On the other hand, with $C(m^2)$ defined in Eq. (11.63), the fermion renormalization constant obtained from Eq. (11.31) is

$$\begin{aligned} Z_\psi &= 1 + C(m^2) - \Sigma_A(m^2) \\ &= 1 + \frac{\alpha}{2\pi}\left[-\frac{1}{2}\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi M^2}{m^2} - 1\right) + \int da\,(1-a)\left(\frac{2(1+a)}{a} + \ln a^2\right)\right], \end{aligned} \tag{11.75}$$

and so we have in total

$$F_1(0) = 1 + \frac{\alpha}{2\pi}\int da\left[2(1-a) + (1-2a)\,\ln a^2\right] = 1\,. \tag{11.76}$$