4 Quantization of the Dirac field

The quantization of the Dirac field proceeds almost along the same lines as that for the classical field, except for one important difference: instead of commutation relations for the fields we will need *anticommutation* relations to ensure a positive spectrum.

Quantized Hamiltonian. To see this, let's calculate the Hamiltonian without imposing any commutation relations yet. We start with the general solutions $\psi(x)$, $\overline{\psi}(x)$ of the Dirac equation, which we reinterpret as operators on a state space:

$$\begin{aligned}\psi(x) &= \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E_p}\sum_s\Big(a_{p,s}\,u_{p,s}\,e^{-ipx}+b^\dagger_{p,s}\,v_{p,s}\,e^{ipx}\Big)_{p^0=E_p}\,,\\ \overline{\psi}(x) &= \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E_p}\sum_s\Big(b_{p,s}\,\bar{v}_{p,s}\,e^{-ipx}+a^\dagger_{p,s}\,\bar{u}_{p,s}\,e^{ipx}\Big)_{p^0=E_p}\,.\end{aligned} \tag{4.1}$$

The coefficients $a_{p,s}=a_s(\boldsymbol{p})$ and $b_{p,s}=b_s(\boldsymbol{p})$ inherit the operator structure, whereas $u_{p,s}=u_s(\boldsymbol{p})$ and $v_{p,s}=v_s(\boldsymbol{p})$ are the Dirac spinors that we worked out above. The conjugate momentum is

$$\Pi=\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=\overline{\psi}\,i\gamma^0=i\psi^\dagger\,, \tag{4.2}$$

and therefore the Hamiltonian becomes

$$H=\int d^3x\,\psi^\dagger(x)\,\gamma^0(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\psi(x)\,. \tag{4.3}$$

This agrees with our earlier result (3.37) extracted from the energy momentum tensor. Note also that $\gamma^0(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)$ is the Dirac Hamiltonian that is well known from quantum mechanics.

When we insert the Fourier decomposition for the fields, then after some calculation (which is analogous to Eqs. (2.6–2.11)) we arrive at

$$H=\int\frac{d^3p}{2E_p}\,E_p\sum_s\Big(a^\dagger_{p,s}\,a_{p,s}-b_{p,s}\,b^\dagger_{p,s}\Big)\,. \tag{4.4}$$

(Ex) The calculation goes along the same lines as before: take the three-dimensional Fourier transform

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\,\widetilde{\psi}_p(t)\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\,,\qquad \widetilde{\psi}_p(t)=\frac{1}{2E_p}\Big(a_{p,s}\,u_{p,s}\,e^{-iE_pt}+b^\dagger_{-p,s}\,v_{-p,s}\,e^{iE_pt}\Big) \tag{4.5}$$

and plug it into the Hamiltonian, which in momentum space becomes

$$H=\int d^3p\,\widetilde{\psi}^\dagger_p(t)\,\gamma^0(\boldsymbol{p}\cdot\boldsymbol{\gamma}+m)\,\widetilde{\psi}_p(t)\,. \tag{4.6}$$

From Eq. (3.54) we know that the solutions of the Dirac equation are eigenfunctions of the Dirac Hamiltonian, which simplifies the calculations a lot:

$$\gamma^0(\boldsymbol{p}\cdot\boldsymbol{\gamma}+m)\,\widetilde{\psi}_p(t)=\frac{1}{2}\Big(a_{p,s}\,u_{p,s}\,e^{-iE_pt}-b^\dagger_{-p,s}\,v_{-p,s}\,e^{iE_pt}\Big)\,. \tag{4.7}$$

With the orthogonality relations (3.61) and (3.63), the time dependencies cancel and one arrives at the result above.

Actually, the result in Eq. (4.4) looks rather suspicious because of the minus sign. Suppose we postulate canonical commutation relations:

$$[\psi_\alpha(x), \psi^\dagger_\beta(y)]_{x^0=y^0} \stackrel{!}{=} \delta_{\alpha\beta}\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \tag{4.8}$$

Then the corresponding commutator relations in momentum space would read

$$[a_{p,s}, a^\dagger_{p',s'}] = [b^\dagger_{p,s}, b_{p',s'}] = 2E_p\, \delta_{ss'}\, \delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \tag{4.9}$$

with all other commutators zero, which is easy to verify by inserting the Fourier decomposition into Eq. (4.8). In that way, once we subtract the vacuum energy, the Hamiltonian is proportional to $a^\dagger a - b^\dagger\, b$ and therefore the energy

$$\langle\lambda|H|\lambda\rangle = \int\!\frac{d^3p}{2E_p}\, E_p\, \langle\lambda|\, a^\dagger_{p,s}\, a_{p,s} - b^\dagger_{p,s}\, b_{p,s}\, |\lambda\rangle \tag{4.10}$$

is unbounded from below. We could ensure that the energy is positive by demanding a negative norm, $||b_{p,s}\,|\lambda\rangle||^2 < 0$, but this violates unitarity. So apparently we face a dilemma: either we have an unstable vacuum (negative energies) or we violate unitarity of the theory (negative norms).

The correct way to resolve the problem is to impose **anticommutation relations:**

$$\{\psi_\alpha(x), \psi^\dagger_\beta(y)\}_{x^0=y^0} = \delta_{\alpha\beta}\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{4.11}$$

with all other *anti*commutators zero, which entails

$$\{a_{p,s}, a^\dagger_{p',s'}\} = \{b_{p,s}, b^\dagger_{p',s'}\} = 2E_p\, \delta_{ss'}\, \delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \tag{4.12}$$

again with all other anticommutators zero. In that case the second term in (4.4) picks up a minus sign, and after throwing away the infinite constant the normal-ordered Hamiltonian is again positive:

$$H = \int \frac{d^3p}{2E_p}\, E_p \sum_s \left(a^\dagger_{p,s}\, a_{p,s} + b^\dagger_{p,s}\, b_{p,s}\right). \tag{4.13}$$

In that way the normal ordering for fermions introduces a minus sign for each interchange of operators. The same result follows for the four-momentum operator:

$$P^\mu = \int \frac{d^3p}{2E_p}\, p^\mu \sum_s \left(a^\dagger_{p,s}\, a_{p,s} + b^\dagger_{p,s}\, b_{p,s}\right). \tag{4.14}$$

Fock space and Fermi-Dirac statistics. Despite the anticommutation relation for the fields, the commutation relations (2.24) for the momentum operator still hold as a consequence of the identity $[AB, C] = A\{B, C\} - \{A, C\}B$. Hence we can take over the analysis from the scalar field: the vacuum is still defined by $a_{p,s}\,|0\rangle = b_{p,s}\,|0\rangle = 0$, multi-particle states are obtained by acting on the vacuum with $a^\dagger_{p,s}$ or $b^\dagger_{p,s}$, and their normalization is the same as before. Note in particular that the norm is positive:[4]

$$\langle 0|\, a_{p,s}\, a^\dagger_{p',s'}\, |0\rangle = \langle 0|\, b_{p,s}\, b^\dagger_{p',s'}\, |0\rangle = 2E_p\, \delta_{ss'}\, \delta^3(\boldsymbol{p}-\boldsymbol{p}')\,. \tag{4.15}$$

[4] Remember that $\delta^3(\mathbf{0})$ is proportional to the volume, so this infrared divergence is not a serious problem. Had we worked with smeared operators from the beginning (at the expense of a simple notation), the norm would be well-defined.

As before, the eigenvalue of the momentum operator P^μ is the total momentum of the state.

However, there is one important difference: since these operators anticommute between themselves, an $N-$particle state is *antisymmetric* under particle exchange:

$$a_{p,s}^\dagger\, a_{q,r}^\dagger\, |0\rangle = -a_{q,r}^\dagger\, a_{p,s}^\dagger\, |0\rangle\,. \tag{4.16}$$

Therefore, spin-$\frac{1}{2}$ particles are fermions, i.e., they obey Fermi-Dirac statistics. In particular, they satisfy the Pauli principle: no two fermionic states of exactly the same quantum numbers are possible, because we can never create more than one particle in the same state:

$$\{a_{p,s}^\dagger, a_{p,s}^\dagger\} = 0 \quad \Rightarrow \quad a_{p,s}^\dagger\, a_{p,s}^\dagger\, |0\rangle = 0\,. \tag{4.17}$$

This is another manifestation of the **spin-statistics theorem:** Lorentz invariance, positive energies, unitarity (=positive norms) and causality together imply that particles with integer spin obey Bose-Einstein statistics, whereas particles with half-odd integer spin obey Fermi-Dirac statistics. By working out the $U(1)$ charge from Eq. (3.44),

$$Q = \int d^3x\, :\psi^\dagger\,\psi: \, = \int \frac{d^3p}{2E_p} \sum_s \left(a_{p,s}^\dagger\, a_{p,s} - b_{p,s}^\dagger\, b_{p,s}\right), \tag{4.18}$$

we arrive at the same interpretation as for the complex scalar field: $a_{p,s}^\dagger$ and $b_{p,s}^\dagger$ create fermions and antifermions, respectively, and the charge equals the number of particles minus antiparticles. Note that the minus sign in Q is also a consequence of the anticommutation relations: Q was non-negative in the classical theory, where it could be interpreted as a scalar product between fields, cf. Eq. (3.73).

The spin operator that follows from the classical Noether charge (5.21) is given by

$$\int d^3x\, :\psi^\dagger\, \frac{\mathbf{\Sigma}}{2}\,\psi: \,. \tag{4.19}$$

One can show (Peskin-Schroeder, p.61) that applying it to a state $a_{p,s}^\dagger\,|0\rangle$ gives eigenvalue $s/2$ whereas applied to $b_{p,s}^\dagger\,|0\rangle$ it gives eigenvalue $-s/2$, where $s=\pm 1$. Therefore, $a_{p,s}^\dagger\,|0\rangle$ describes a fermion (for example an electron) with mass m, energy E_p, spin $\frac{1}{2}$ and spin polarization $s/2$, whereas $b_{p,s}^\dagger\,|0\rangle$ describes an antifermion (positron) with mass m, energy E_p, spin $\frac{1}{2}$ and spin polarization $-s/2$. The state $\overline{\psi}(x)\,|0\rangle$ describes a fermion at position x and $\psi(x)\,|0\rangle$ an antifermion at position x.

Causality. Despite the anticommutator relations that we imposed for the Dirac fields, the microcausality axiom must remain unchanged: all physical observables are bosonic operators and must commute at spacelike distances,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] \stackrel{!}{=} 0 \qquad \text{if} \quad (x-y)^2 < 0\,. \tag{4.20}$$

This is ensured by requiring

$$S_{\alpha\beta}(x-y) := \{\psi_\alpha(x), \overline{\psi}_\beta(y)\} \stackrel{!}{=} 0 \qquad \text{if} \quad (x-y)^2 < 0\,, \tag{4.21}$$

(Ex) which is the generalization of Eq. (2.74) in the scalar case. Eq. (4.20) can be checked directly for fermion bilinears $\mathcal{O}_i(x) = \overline{\psi}(x)\,\Gamma_i\,\psi(x)$, where Γ_i is any of the Dirac matrices in Eq. (3.21), by exploiting the identity

$$[AB, CD] = A\,\{B,C\}\,D - C\,\{A,D\}\,B - \frac{\{A,C\}\,[B,D] + [A,C]\,\{B,D\}}{2}\,. \tag{4.22}$$

Inserting the Fourier decomposition (4.1), the anticommutator relation (4.12) and the completeness relations (3.62), this expression becomes

$$\begin{aligned} S(z) &= \frac{1}{(2\pi)^3}\int\frac{d^3p}{2E_p}\left((\not p + m)\,e^{-ipz} + (\not p - m)\,e^{ipz}\right) \\ &= (i\not\partial + m)\,\frac{1}{(2\pi)^3}\int\frac{d^3p}{2E_p}\left(e^{-ipz} - e^{ipz}\right) = (i\not\partial + m)\,\Delta(z) \end{aligned} \tag{4.23}$$

where $\Delta(z)$ is the scalar analogue in Eq. (2.69). From here it is easy to recover our original commutator relations (4.11):

$$S(z)\big|_{z^0=0} = (i\not\partial + m)\,\Delta(z)\big|_{z^0=0} = \gamma^0\,\delta^3(\boldsymbol{z})\,, \tag{4.24}$$

because $\partial_0\,\Delta(z)\big|_{z^0=0} = -i\delta^3(\boldsymbol{z})$, $\partial_i\,\Delta(z)\big|_{z^0=0} = 0$ and $\Delta(z)\big|_{z^0=0} = 0$.

Feynman propagator. Similarly, we define the Feynman propagator for fermions as

$$S_F(x-y) := \langle 0|\,\mathsf{T}\,\psi(x)\overline{\psi}(y)\,|0\rangle = \begin{cases} \langle 0|\,\psi(x)\overline{\psi}(y)\,|0\rangle & \text{if} \quad x^0 \geq y^0\,, \\ -\langle 0|\,\overline{\psi}(y)\psi(x)\,|0\rangle & \text{if} \quad y^0 \geq x^0\,, \end{cases} \tag{4.25}$$

with the crucial difference of the minus sign. It is necessary because if $(x-y)^2 < 0$ we have $S(x-y) = 0$ and therefore $\psi(x)\,\overline{\psi}(y) = -\overline{\psi}(y)\,\psi(x)$. For spacelike distances the question of whether $x^0 > y^0$ or $x^0 < y^0$ depends on the frame, and to arrive at a frame-independent definition of the time-ordering symbol T the expression for $\mathsf{T}\,\psi(x)\,\overline{\psi}(y)$ for $x^0 > y^0$ and $x^0 < y^0$ must agree.

Using the definition above and inserting the Fourier decomposition, one evaluates $S_F(z) = (i\not\partial + m)\,\Delta_F(z)$ and therefore the fermion propagator becomes

$$S_F(z) = \int\frac{d^4p}{(2\pi)^4}\,e^{-ipz}\,\frac{i\,(\not p + m)}{p^2 - m^2 + i\epsilon}\,. \tag{4.26}$$

Since $(\not p + m)\,(\not p - m) = p^2 - m^2$, the inverse propagator in momentum space has the form

$$S_F^{-1}(p) = -i\,(\not p - m)\,. \tag{4.27}$$

The Feynman propagator is a Green function of the Dirac equation, i.e., it is one of the four possible solutions to the equation $(i\not\partial - m)\,G(z) = i\delta^4(z)$. Their interpretation and closure procedure in the complex plane are as in the scalar theory.

Parity. Earlier we have seen that the parity operation $x \to x' = (t, -\boldsymbol{x})$ exchanges the left- and right-handed Weyl spinors:

$$\begin{pmatrix} \psi'_L(x') \\ \psi'_R(x') \end{pmatrix} = \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} \psi'(x') &= \gamma^0\,\psi(x) \\ \overline{\psi}'(x') &= \overline{\psi}(x)\,\gamma^0 \,. \end{aligned} \tag{4.28}$$

Consequently, the bilinears $\overline{\psi}\psi$ and $\overline{\psi}\,i\gamma_5\,\psi$ transform as scalars and pseudoscalars under parity (γ^0 anticommutes with γ_5):

$$\overline{\psi}'(x')\,\psi'(x') = \overline{\psi}(x)\,\psi(x)\,, \qquad \overline{\psi}'(x')\,i\gamma_5\,\psi'(x') = -\overline{\psi}(x)\,i\gamma_5\,\psi(x)\,. \tag{4.29}$$

The factor i is necessary to make the pseudoscalar bilinear real: $(\overline{\psi}\,i\gamma_5\psi)^\dagger = \overline{\psi}\,i\gamma_5\psi$. Likewise, $\overline{\psi}\gamma^\mu\psi$ and $\overline{\psi}\gamma^\mu\gamma_5\psi$ transform as vectors and axialvectors, respectively:

$$\overline{\psi}'(x')\gamma^\mu\psi'(x') = \pm\overline{\psi}(x)\gamma^\mu\psi(x) \qquad \overline{\psi}'(x')\gamma^\mu\gamma_5\psi'(x') = \mp\overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)\,, \tag{4.30}$$

where the upper sign corresponds to $\mu = 0$ and the lower one to $\mu = 1, 2, 3$.

How does parity act on the Fock space? If we introduce the unitary operator U_P that transforms a state as $|\lambda'\rangle = U_P\,|\lambda\rangle$, then the quantum version of Eq. (4.28) follows from the same reasoning as in Eq. (2.59):

$$U_P\,\psi(x)\,U_P^{-1} = \gamma^0\,\psi(x')\,, \qquad U_P\,\overline{\psi}(x)\,U_P^{-1} = \overline{\psi}(x')\,\gamma^0\,. \tag{4.31}$$

We ignore possible phase factors for simplicity because they are not important for the discussion. Applied to the Fourier decomposition (4.1), we can work out the action of U_P on the creation and annihilation operators:

$$U_P\,a_{p,s}\,U_P^{-1} = a_{-p,s}\,, \qquad U_P\,b^\dagger_{p,s}\,U_P^{-1} = -b^\dagger_{-p,s}\,. \tag{4.32}$$

(Ex) To derive this, start with

$$\gamma^0\psi(x') = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E_p}\sum_s\Big(a_{p,s}\,(\gamma^0 u_{p,s})\,e^{-iE_pt-\boldsymbol{p}\cdot\boldsymbol{x}} + b^\dagger_{p,s}\,(\gamma^0 v_{p,s})\,e^{iE_pt+\boldsymbol{p}\cdot\boldsymbol{x}}\Big)\,.$$

From Eq. (3.59) it follows that $\gamma^0 u_{p,s} = u_{-p,s}$ and $\gamma^0 v_{p,s} = -v_{-p,s}$; remember our shorthand notation $u_{p,s} = u_s(\boldsymbol{p})$, so the minus sign switches only the spatial components. Exchanging $\boldsymbol{p} \to -\boldsymbol{p}$ in the integral leads to

$$\gamma^0\psi(x') = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E_p}\sum_s\Big(a_{-p,s}\,u_{p,s}\,e^{-ipx} - b^\dagger_{-p,s}\,v_{p,s}\,e^{ipx}\Big)_{p^0=E_p}\,, \tag{4.33}$$

and comparison with the direct expression for $U_P\,\psi(x)\,U_P^{-1}$ gives the result in Eq. (4.32).

Applied to one-particle and -antiparticle states, this entails

$$\begin{aligned} U_P\,|\boldsymbol{p},s,a\rangle &= U_P\,a^\dagger_{p,s}|0\rangle = a^\dagger_{-p,s}|0\rangle = |-\boldsymbol{p},s,a\rangle\,, \\ U_P\,|\boldsymbol{p},s,b\rangle &= U_P\,b^\dagger_{p,s}|0\rangle = -b^\dagger_{-p,s}|0\rangle = -|-\boldsymbol{p},s,b\rangle\,, \end{aligned} \tag{4.34}$$

where we assumed parity invariance of the vacuum $U_P\,|0\rangle = |0\rangle$. The relative minus sign tells us that fermions and antifermions carry opposite **intrinsic parity**. For scalar fields we would not get the relative minus sign: the intrinsic parity of a spin-0 particle and its antiparticle are equal.

Charge conjugation. As we remarked in the context of Majorana spinors, one cannot construct a charge-conjugate Dirac spinor in the form $\psi \to \psi^*$ because this is not Lorentz-invariant: since $D^*(\Lambda) \neq D(\Lambda)$, a Lorentz transformation will mix ψ and ψ^*. Instead, the property $\gamma^{\mu *} = \gamma^2\gamma^\mu\gamma^2$ implies $D^*(\Lambda) = -\gamma^2\, D(\Lambda)\,\gamma^2$, which allows us to define the operation of **charge conjugation** as

$$\psi^c = -i\gamma^2\,\psi^*\,, \qquad \overline{\psi}^c = i\overline{\psi}^*\gamma^2\,. \tag{4.35}$$

This is now indeed compatible with a Lorentz transformation:

$$(\psi^c)'(x') = -i\gamma^2\left(\psi'(x')\right)^* = -i\gamma^2\,D^*(\Lambda)\,\psi^*(x) = D(\Lambda)\,\psi^c(x)\,. \tag{4.36}$$

Let's work this out in the chiral representation: **(Ex)**

$$\gamma^2 D(\Lambda)^*\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0\end{pmatrix}\begin{pmatrix} D_L^*(\Lambda) & 0 \\ 0 & D_R^*(\Lambda)\end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0\end{pmatrix} = \begin{pmatrix} -\sigma^2 D_R^*(\Lambda)\,\sigma^2 & 0 \\ 0 & -\sigma^2 D_L^*(\Lambda)\,\sigma^2\end{pmatrix}.$$

Using the explicit form of $D_{L,R}(\Lambda)$ from Eq. (3.9) together with the properties $\sigma^2\sigma^i\sigma^2 = -\sigma^{i*}$ and $\sigma^2\sigma^2 = 1$, it follows that

$$\sigma^2 D_{L,R}^*(\Lambda)\,\sigma^2 = D(\Lambda)_{R,L} \qquad \Rightarrow \qquad \gamma^2 D^*(\Lambda)\gamma^2 = -D(\Lambda)\,. \tag{4.37}$$

In terms of Weyl spinors, the charge-conjugate spinor takes the form

$$\psi^c = \begin{pmatrix} \psi_L^c \\ \psi_R^c \end{pmatrix} = \begin{pmatrix} -i\sigma^2\,\psi_R^* \\ i\sigma^2\,\psi_L^* \end{pmatrix}. \tag{4.38}$$

Let's express ψ^* through the conjugate spinor: $\psi^* = (\psi^\dagger)^T = (\overline{\psi}\gamma^0)^T = \gamma^0\,\overline{\psi}^T$. Defining the **charge-conjugation matrix** $C = i\gamma^2\gamma^0$, we arrive at

$$\psi^c = C^T\,\overline{\psi}^T\,, \qquad \overline{\psi}^c = \psi^T C^T\,. \tag{4.39}$$

The transpose on a spinor is not really necessary; it just means that $\psi^c_\alpha = (C^T)_{\alpha\beta}\,\overline{\psi}_\beta = \overline{\psi}_\beta\, C_{\beta\alpha}$. The charge-conjugation matrix has some useful properties:

$$C^\dagger = C^T = C^{-1} = -C, \qquad C\,\gamma_5^T C^T = \gamma_5, \qquad C\,\gamma_\mu^T\, C^T = -\gamma_\mu\,. \tag{4.40}$$

Since charge conjugation does not change the spacetime argument, we can identify it directly with the operator transformation:

$$U_C\,\psi\,U_C^{-1} = C^T\,\overline{\psi}^T\,, \qquad U_C\,\overline{\psi}\,U_C^{-1} = \psi^T C^T\,. \tag{4.41}$$

If we insert the Fourier decomposition and use the relations $\gamma^2 u_{p,s} = v_{p,s}$ and $\gamma^2 v_{p,s} = u_{p,s}$, which follow again from Eq. (3.59), we arrive at

$$U_C\,a_{p,s}\,U_C^{-1} = b_{p,s}\,, \qquad U_C\,b_{p,s}\,U_C^{-1} = a_{p,s}\,. \tag{4.42}$$

As desired, charge conjugation transforms a particle $|\boldsymbol{p},s,a\rangle$ into its antiparticle $|\boldsymbol{p},s,b\rangle$. Recall that the state $|\boldsymbol{p},s,a\rangle$ describes a particle with spin polarization $s/2$ and the state $|\boldsymbol{p},s,b\rangle$ an antiparticle with spin polarization $-s/2$; therefore, charge conjugation also reverses the helicity.

Time reversal. Although the time reversal operation $x \to x' = (-t, \boldsymbol{x})$ looks similar to the parity transformation, it is probably the most confusing of the discrete symmetries and has a rather special status. In the classical theory, all particles of the time-mirrored system follow their trajectories backwards: the momenta and angular momenta are reversed, and the roles of the initial and final configurations are interchanged. A Dirac spinor transforms as

$$\psi'(x') = \gamma^0\gamma^5\,\psi^c(x) = \gamma^1\gamma^3\,\psi^*(x)\,, \tag{4.43}$$

which can be derived from the transformation behavior of the Dirac equation, or that of fermion bilinears. The need for complex conjugation can be understood intuitively from the picture of antiparticles as particles moving backwards in time (a time reversal of the phase e^{-iE_pt} would lead to negative energies of the mirrored system and necessitates a sign change of i). Correspondingly, the Weyl spinors transform as $\psi'_{L,R}(x') = i\sigma^2\,\psi^*_{L,R}(x)$.

The speciality of time reversal is that, when taking matrix elements, it exchanges the in and out states:

$$\langle U_T\lambda_1|\,\psi_\alpha(x')\,|U_T\lambda_2\rangle = (\gamma^1\gamma^3)_{\alpha\beta}\,\langle\lambda_1|\,\psi_\beta(x)\,|\lambda_2\rangle^* = (\gamma^1\gamma^3)_{\alpha\beta}\,\langle\lambda_2|\,\psi^\dagger_\beta(x)\,|\lambda_1\rangle\,, \tag{4.44}$$

and therefore we cannot simply compare both sides of the equation anymore to obtain a transformation law for the field operators. To do so, we must identify U_T with an **antiunitary** operator, which leads to

$$U_T\,\psi(x)\,U_T^{-1} = \gamma^1\gamma^3\,\psi(x')\,, \qquad U_T\,\overline{\psi}(x)\,U_T^{-1} = \overline{\psi}(x')\,\gamma^3\gamma^1\,, \tag{4.45}$$

again ignoring possible phases. This is compatible with the **Wigner theorem**, which states that symmetries in the quantum theory must be implemented by unitary or antiunitary operators. Note that an antiunitary operator induces complex conjugation for numbers: $U_T\,c\,U_T^{-1} = c^*$. However, since the transformation of the quantum fields ψ, $\overline{\psi}$ no longer requires complex conjugation, the transformation does not send particles to antiparticles but rather particles to *particles*.

The point is that Hilbert state vectors that differ only by phases are physically equivalent, which is why it is sufficient to demand $|\langle U\lambda_1\,|\,U\lambda_2\rangle| = |\langle\lambda_1|\lambda_2\rangle|$ for symmetry operations. This can be realized by a unitary operator,

$$\langle U\lambda_1|\,U\lambda_2\rangle = \langle\lambda_1|\lambda_2\rangle\,, \qquad U(c_1\,|\lambda_1\rangle + c_2\,|\lambda_2\rangle) = c_1\,U\,|\lambda_1\rangle + c_2\,U\,|\lambda_2\rangle \tag{4.46}$$

or an antiunitary operator:

$$\langle U\lambda_1|\,U\lambda_2\rangle = \langle\lambda_1|\lambda_2\rangle^* = \langle\lambda_2|\lambda_1\rangle\,, \qquad U(c_1\,|\lambda_1\rangle + c_2\,|\lambda_2\rangle) = c_1^*\,U\,|\lambda_1\rangle + c_2^*\,U\,|\lambda_2\rangle\,. \tag{4.47}$$

Clearly, both possibilities are compatible with the symmetry requirement, but the essence of the Wigner theorem (whose proof is rather lengthy) is that these are the *only* options. Note that in both cases $U^\dagger U = UU^\dagger = 1$, but the definition of the hermitian conjugate changes in the antiunitary case: $\langle\lambda_1|\,U^\dagger\lambda_2\rangle = \langle\lambda_2|\,U\lambda_1\rangle$. Hence, Eq. (4.44) requires U_T to be antiunitary:

$$\langle U_T\lambda_1|\,\psi(x')\,|U_T\lambda_2\rangle = \langle\lambda_2|\,U_T^\dagger\,\psi(x')^\dagger\,U_T\,|\lambda_1\rangle\,, \tag{4.48}$$

and the comparison with the r.h.s. leads to Eq. (4.45) (again, up to an irrelevant phase factor).

		C	P	T	CPT
S	$\bar{\psi}\psi$	1	1	1	1
P	$\bar{\psi}\, i\gamma_5\, \psi$	1	-1	-1	1
V	$\bar{\psi}\,\gamma^\mu\psi$	-1	$(1,-\mathbf{1})$	$(1,-\mathbf{1})$	-1
A	$\bar{\psi}\,\gamma^\mu\gamma_5\psi$	1	$(-1,\mathbf{1})$	$(1,-\mathbf{1})$	-1
T	$\bar{\psi}\,\sigma^{\mu\nu}\psi$	-1	$\begin{pmatrix} 1 & -\mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix}$	$\begin{pmatrix} -1 & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}$	1
	∂_μ	1	$(1,-\mathbf{1})$	$(-1,\mathbf{1})$	-1

TABLE II.1: Transformation properties under C, P and T.

CPT. The transformation properties of the various fermion bilinears $:\bar{\psi}(x)\,\Gamma\,\psi(x):$ under C, P and T are summarized in Table II.1. The free Dirac action is invariant under C, P and T separately. We can construct more general actions that violate any of these symmetries, but since they must be Lorentz scalars, the free Lorentz indices in γ^μ, $\gamma^\mu\gamma_5$ and $\sigma^{\mu\nu}$ must be contracted with the derivative ∂_μ (or other bilinears). As a consequence, the combined symmetry CPT is always conserved: one cannot build a Lorentz-invariant quantum field theory with a hermitian Hamiltonian that violates CPT.

For example, under charge conjugation the bilinears behave as

$$\begin{aligned} \bar{\psi}\,\Gamma\,\psi \;\to\; U_C\,(\bar{\psi}\,\Gamma\,\psi)\,U_C^{-1} &= \psi^T C^T\Gamma\, C^T\,\bar{\psi}^T = (\psi)_\alpha\,(C^T\Gamma\, C^T)_{\alpha\beta}\,\bar{\psi}_\beta \\ &= -\bar{\psi}_\beta\,(C^T\Gamma\, C^T)_{\alpha\beta}\,(\psi)_\alpha = \bar{\psi}\,(C\,\Gamma^T C^T)\,\psi\,, \end{aligned} \tag{4.49}$$

where we used fermion anticommutation: $\psi_\alpha\bar{\psi}_\alpha = -\bar{\psi}_\alpha\psi_\alpha$ (the infinite constant vanishes by normal ordering). Together with the relations (4.40) it is then straightforward to obtain the 'C' column in Table II.1; note that the vector and tensor bilinears switch sign under charge conjugation. Similarly, under parity one has

$$\bar{\psi}(x)\,\Gamma\,\psi(x) \;\to\; U_P\,\big(\bar{\psi}(x)\,\Gamma\,\psi(x)\big)\,U_P^{-1} = \bar{\psi}(x')\,(\gamma^0\,\Gamma\,\gamma^0)\,\psi(x')\,, \tag{4.50}$$

and time reversal leads to

$$\bar{\psi}(x)\,\Gamma\,\psi(x) \;\to\; U_T\,\big(\bar{\psi}(x)\,\Gamma\,\psi(x)\big)\,U_T^{-1} = \bar{\psi}(x')\,(\gamma^3\gamma^1\,\Gamma^*\,\gamma^1\gamma^3)\,\psi(x')\,, \tag{4.51}$$

where the complex conjugate Γ^* is a consequence of the antiunitarity: $U_T\,\Gamma\,U_T^{-1} = \Gamma^*$. Since we can express time reversal through charge conjugation via Eq. (4.43), the result can be also written as

$$U_T\,\big(\bar{\psi}(x)\,\Gamma\,\psi(x)\big)\,U_T^{-1} = \bar{\psi}(x')\,\gamma_5\gamma^0\,(C\,\Gamma^T C^T)\,\gamma^0\gamma_5\,\psi(x')\,. \tag{4.52}$$

In summary, the signs in the table are simply obtained from the signs of

$$C \;\to\; C\,\Gamma^T C^T\,, \qquad P \;\to\; \gamma^0\,\Gamma\,\gamma^0\,, \qquad T \;\to\; \gamma_5\gamma^0\,(C\,\Gamma^T C^T)\,\gamma^0\gamma_5\,. \tag{4.53}$$

Taking everything in combination, the CPT symmetry amounts to $\gamma_5\,\Gamma\gamma_5$ together with a spacetime reflection $x \to -x$.