Appendix C

Euclidean conventions

When employing a Minkowski metric (which is what we use throughout the main text), one must be mindful of the $i\epsilon$ prescription that is necessary to make many relations in QFT well-defined. It arises from the imaginary-time boundary conditions (2.2.21), which lead to boundary conditions on d^4x and d^4p integrals. An alternative is to define $x^4 = ix_0$ and $p^4 = ip_0$ and perform a **Wick rotation** to write

$$\begin{aligned} \int d^4x &= \int d^3x \int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dx_0 = -i \int d^3x \int_{-\infty}^{\infty} dx_4 \,, \\ \int d^4p &= \int d^3p \int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dp_0 = i \int d^3p \int_{-\infty}^{\infty} dp_4 \,. \end{aligned} \tag{C.1}$$

Note that the integration paths in x_0 and p_0 rotate in opposite directions and thus

$$i \int d^4x = \int d^4x_E \qquad \text{but} \qquad \int d^4p = i \int d^4p_E \,. \tag{C.2}$$

Since $x^2 = x_0^2 - \boldsymbol{x}^2 = -\boldsymbol{x}^2 - x_4^2 = -x_E^2$, this amounts to using a **Euclidean metric** with signature $(+,+,+,+)$.

Euclidean conventions. In general, we define Euclidean vectors a_E^μ and tensors $T_E^{\mu\nu}$ such that their spatial parts agree with Minkowski space:

$$a_E^\mu = \begin{bmatrix} \boldsymbol{a} \\ ia_0 \end{bmatrix}, \quad T_E^{\mu\nu} = \begin{bmatrix} T^{ij} & iT^{i0} \\ iT^{0i} & -T^{00} \end{bmatrix}, \tag{C.3}$$

where 'E' stands for Euclidean and no subscript refers to the Minkowski quantity. As a consequence, the Lorentz-invariant scalar product of any two four-vectors differs by a minus sign from its Minkowski counterpart:

$$a_E \cdot b_E = \sum_{k=1}^{4} a_E^k \, b_E^k = -a \cdot b \,. \tag{C.4}$$

Therefore, a vector is spacelike if $a^2 > 0$ and timelike if $a^2 < 0$. Because the Euclidean metric is positive, we can drop the distinction between upper and lower indices.

To preserve the meaning of the slash $\not{a} = a^0\gamma^0 - \boldsymbol{a}\cdot\boldsymbol{\gamma}$, we must also redefine the $\gamma-$matrices:

$$i\gamma_E^\mu = \begin{bmatrix} \boldsymbol{\gamma} \\ i\gamma_0 \end{bmatrix}, \quad \gamma_E^5 = \gamma^5 \quad \Rightarrow \quad \not{a}_E = a_E\cdot\gamma_E = i\not{a}, \quad \{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu}\,. \tag{C.5}$$

Our sign convention for the Euclidean $\gamma-$matrices changes all signs in the Clifford algebra relation to be positive, and since this implies $(\gamma_E^i)^2 = 1$ for $i = 1\dots 4$ we can choose them to be hermitian: $\gamma_E^\mu = \left(\gamma_E^\mu\right)^\dagger$. In the standard representation they read

$$\gamma_E^k = \begin{bmatrix} 0 & -i\tau_k \\ i\tau_k & 0 \end{bmatrix}, \quad \gamma_E^4 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix},$$

where the τ_k are the usual Pauli matrices from Eq. (A.1.4). Also the generators of the Clifford algebra are then hermitian, with $(\sigma_E^{\mu\nu})^\dagger = \sigma_E^{\mu\nu}$:

$$\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu] \quad \Rightarrow \quad \sigma_E^{\mu\nu} = -\frac{i}{2}\,[\gamma_E^\mu, \gamma_E^\nu]\,. \tag{C.6}$$

Despite appearances, this does not alter the Lorentz transformation properties and the definition of the conjugate spinor as $\overline{\psi} = \psi^\dagger\gamma^4$ (which was necessary to make a bilinear $\overline{\psi}\psi$ Lorentz-invariant) remains intact. Denoting the representation matrix $\psi'(x') = D(\Lambda)\,\psi(x)$ of the Lorentz transformation by

$$D(\Lambda) = \exp\left[-\frac{i}{4}\,\omega_{\mu\nu}\,\sigma^{\mu\nu}\right] = \exp\left[-\frac{i}{4}\,\omega_E^{\mu\nu}\,\sigma_E^{\mu\nu}\right], \tag{C.7}$$

then irrespective of $\gamma^4\,(\sigma_E^{\mu\nu})^\dagger\,\gamma^4 \neq \sigma_E^{\mu\nu}$ the relation $\gamma^4\,D(\Lambda)^\dagger\,\gamma^4 = D(\Lambda)^{-1}$ still holds, because the infinitesimal Lorentz transformation $\omega_E^{\mu\nu}$ which is related to its Minkowski counterpart via (C.3) is now complex. Hence

$$\overline{\psi}'(x') = \psi^\dagger(x)\,D(\Lambda)^\dagger\,\gamma^4 = \psi^\dagger(x)\,\gamma^4\,D(\Lambda)^{-1} = \overline{\psi}\,D(\Lambda)^{-1}\,, \tag{C.8}$$

and therefore $\overline{\psi}\psi$ is Lorentz-invariant, $\overline{\psi}\gamma_E^\mu\psi$ transforms like a Lorentz vector, etc.

For derivatives, Eq. (C.3) implies

$$\partial_\mu^E = \begin{bmatrix} \boldsymbol{\nabla} \\ -i\partial_0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{aligned} \partial\cdot a &= \partial_0\,a^0 + \nabla\cdot\boldsymbol{a} = (\partial\cdot a)_E\,, \\ \not{\partial} &= \gamma^0\partial_0 + \boldsymbol{\gamma}\cdot\nabla = i\not{\partial}_E\,, \\ \Box &= \partial_0^2 - \nabla^2 = -\Box_E\,. \end{aligned} \tag{C.9}$$

As a result, a fermionic action becomes

$$e^{iS} = \exp\left[i\int d^4x\,\overline{\psi}\,(i\not{\partial} - m)\,\psi\right] = \exp\left[-\int d^4x_E\,\overline{\psi}\,(\not{\partial}_E + m)\,\psi\right] = e^{-S_E}\,. \tag{C.10}$$

In this way, the **Euclidean action** S_E is non-negative and the term e^{-S_E} defines a probability measure in the path integral formulation.

Another advantage of the Euclidean metric is that one can perform numerical calculations directly in a given frame (e.g. using `Mathematica`), with explicit $\gamma-$matrices and without the need for inserting the metric tensor in each summation. To transform an expression from Minkowski to Euclidean space, it is usually sufficient to employ the replacement rules collected in Table C.1 which can be read off from the spatial components.

Minkowski	Euclidean
$a\cdot b$	$-a\cdot b$
a^μ	a^μ
γ^μ	$i\gamma^\mu$
γ_5	γ_5
$\not{a}$	$-i\not{a}$
$g^{\mu\nu}$	$-\delta^{\mu\nu}$
$a^\mu b^\nu$	$a^\mu b^\nu$
∂_μ	∂^μ

Minkowski	Euclidean
$[\gamma^\mu,\gamma^\nu]$	$-[\gamma^\mu,\gamma^\nu]$
$[\gamma^\mu,\not{a}]$	$[\gamma^\mu,\not{a}]$
$[\gamma^\mu,\gamma^\nu,\not{a}]$	$i[\gamma^\mu,\gamma^\nu,\not{a}]$
$[\not{a},\not{b}]$	$-[\not{a},\not{b}]$
$\varepsilon^{\mu\nu\rho\alpha}a_\alpha$	$i\varepsilon^{\mu\nu\rho\alpha}a^\alpha$
$\varepsilon^{\mu\nu\alpha\beta}a_\alpha\, b_\beta$	$i\varepsilon^{\mu\nu\alpha\beta}a^\alpha b^\beta$
$\varepsilon^{\mu\alpha\beta\gamma}a_\alpha\, b_\beta\, c_\gamma$	$i\varepsilon^{\mu\alpha\beta\gamma}a^\alpha b^\beta c^\gamma$
$\varepsilon^{\mu\nu\alpha\beta}a_\alpha\gamma_\beta$	$-\varepsilon^{\mu\nu\alpha\beta}a^\alpha\gamma^\beta$

Table C.1: Replacement rules for some frequently occurring quantities. For expressions with Lorentz indices, the right columns define their Euclidean version in the sense of Eqs. (C.3) and (C.5). Each additional Minkowski summation over Lorentz indices leads to a minus sign in Euclidean conventions.

Expressions involving $\varepsilon^{\mu\nu\alpha\beta}$ work along the same lines: the spatial parts of Lorentz tensors are identical in Minkowski and Euclidean conventions, so this must also hold for $\varepsilon^{\mu\nu\alpha\beta}a_\alpha b_\beta$. In Euclidean space the $\varepsilon-$tensor is defined by $\varepsilon_{1234}=\varepsilon^{1234}=1$, whereas in Minkowski conventions one has $\varepsilon_{0123}=-\varepsilon^{0123}=1$, i.e., the spatial components switch sign when lowering or raising indices. Denoting spatial indices by i,j,k and summing over k, one has

$$\begin{aligned}\varepsilon^{ij\alpha\beta}a_\alpha b_\beta &= \varepsilon^{ijk0}\,(a_k b_0 - a_0 b_k) = -\varepsilon^{ijk0}\,(a^k b^0 - a^0 b^k)\\ &= i\varepsilon^{ijk4}\,(a^k b^4 - a^4 b^k)_E = \left(i\varepsilon^{ij\alpha\beta}a^\alpha b^\beta\right)_E,\end{aligned} \tag{C.11}$$

because $\varepsilon^{1234}=1=\varepsilon^{1230}$ and $a^0=-ia^4_E$. Repeating this for rank-1 and rank-3 tensors results in the identities in Table C.1 (which would also follow from Eq. (C.29) below).

Euclidean Feynman rules. We now drop the index 'E' and write all subsequent formulas in Euclidean space. The Euclidean action of QCD is

$$S=\int d^4x\,\left[\overline{\psi}\,(\not{D}+\mathsf{M})\,\psi+\tfrac{1}{4}F^a_{\mu\nu}\,F^{\mu\nu}_a\right], \tag{C.12}$$

where $D_\mu=\partial_\mu+igA_\mu$ (this is consistent with Eq. (2.1.3) because the spatial component is $\boldsymbol{D}=\nabla+ig\boldsymbol{A}$). As a consequence,

$$F_{\mu\nu}(x)=-\partial_\mu A_\nu+\partial_\nu A_\mu-ig\,[A_\mu,A_\nu]\,. \tag{C.13}$$

The Fourier transform is *defined* in Euclidean space,

$$F(x)=\int\frac{d^dp}{(2\pi)^d}\,e^{ip\cdot x}\,F(p)\,,\qquad F(p)=\int d^4x\,e^{-ip\cdot x}\,F(x)\,, \tag{C.14}$$

which would technically lead to i factors in front of the integrals in Minkowski space, but this can be compensated by removing factors of i from the propagators and vertices in momentum space.

The resulting **Feynman rules** are obtained by taking the Feynman rules in Minkowski space, transforming to Euclidean space, and splitting off a factor i from each 1PI quantity (vertices and inverse propagators). This yields:

- Quark propagator:

$$S_0^{-1}(p) = Z_\psi \left(i\not{p} + m_B\right), \qquad S_0(p) = \frac{1}{Z_\psi}\,\frac{-i\not{p} + m_B}{p^2 + m_B^2} \tag{C.15}$$

- Gluon propagator (we redefine $T_q^{\mu\nu} = \delta^{\mu\nu} - q^\mu q^\nu/q^2$ and $L_q^{\mu\nu} = q^\mu q^\nu/q^2$):

$$(D_0^{-1})^{\mu\nu}(q) = q^2 \left(Z_A\, T_q^{\mu\nu} + \frac{1}{\xi}\, L_q^{\mu\nu} \right), \quad D_0^{\mu\nu}(q) = \frac{Z(q^2)\, T_q^{\mu\nu} + \xi\, L_q^{\mu\nu}}{q^2} \tag{C.16}$$

- Ghost propagator:

$$D_{G,0}^{-1}(q) = -Z_c\, q^2\,, \qquad D_{G,0}(q) = -\frac{1}{Z_c\, q^2} \tag{C.17}$$

- Quark-gluon vertex:

$$\Gamma_0^\mu = ig\, \mathsf{t}_a\, Z_\Gamma\, \gamma^\mu \tag{C.18}$$

- Ghost-gluon vertex:

$$\Gamma_{\mathrm{gh},0}^\mu(p) = -igf_{abc}\, \widetilde{Z}_\Gamma\, p^\mu \tag{C.19}$$

- Three-gluon vertex:

$$\begin{aligned}\Gamma_{3g,0}^{\mu\nu\rho}(p_1,p_2,p_3) = igf_{abc}\, Z_{3g}\, \big[&(p_1-p_2)^\rho\, \delta^{\mu\nu} \\ &+(p_2-p_3)^\mu\, \delta^{\nu\rho} + (p_3-p_1)^\nu\, \delta^{\rho\mu}\big]\,.\end{aligned} \tag{C.20}$$

- Four-gluon vertex:

$$\begin{aligned}\Gamma_{4g,0}^{\mu\nu\rho\sigma} = -g^2 Z_{4g} \Big[\; & f_{abe}f_{cde}\,(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\nu\rho}\delta^{\mu\sigma}) \\ + \; & f_{ace}f_{bde}\,(\delta^{\mu\nu}\delta^{\rho\sigma} - \delta^{\nu\rho}\delta^{\mu\sigma}) \\ + \; & f_{ade}f_{cbe}\,(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma}) \Big]\,.\end{aligned} \tag{C.21}$$

The Lorentz-invariant dressing functions are identical except that the arguments pick up minus signs. This is often indicated by capital letters such as $Q^2 = q_E^2 = -q_M^2$. In general, if one defines

$$\{s_1,\, s_2,\, s_3,\, \dots\} = \{p_E^2,\, q_E^2,\, p_E\cdot q_E,\, \dots\} = \{-p_M^2,\, -q_M^2,\, -p_M\cdot q_M,\, \dots\}\,, \tag{C.22}$$

then the quantities $F(s_1,s_2,s_3,\dots)$ are the same in Euclidean and Minkowski space. Also for this reason it is convenient to break down Lorentz-*covariant* relations to Lorentz-*invariant* relations, because then the transformations from Minkowski to Euclidean space and vice versa become trivial (assuming that the correct integration paths for loop integrals are chosen such as in Fig. 2.7).

Euclidean formulas. We suppress again the index 'E' and collect some useful Euclidean formulas. The γ_5 matrix is defined by

$$\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4 = -\frac{1}{24}\,\varepsilon^{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{C.23}$$

with $\varepsilon^{1234} = 1$. It is convenient to define the fully antisymmetric combinations of Dirac matrices by the commutators

$$[A,B] = AB - BA\,, \tag{C.24}$$

$$[A,B,C] = [A,B]\,C + [B,C]\,A + [C,A]\,B\,, \tag{C.25}$$

$$[A,B,C,D] = [A,B,C]\,D + [B,C,D]\,A + [C,D,A]\,B + [D,A,B]\,C\,. \tag{C.26}$$

Inserting $\gamma-$matrices, this yields the antisymmetric combinations

$$[\gamma^\mu,\gamma^\nu] = \gamma_5\,\varepsilon^{\mu\nu\alpha\beta}\,\gamma^\alpha\gamma^\beta\,, \tag{C.27}$$

$$\tfrac{1}{6}\,[\gamma^\mu,\gamma^\nu,\gamma^\rho] = \tfrac{1}{2}\,(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu\gamma^\mu) = \tfrac{1}{4}\,\{[\gamma^\mu,\gamma^\nu],\gamma^\rho\} = -\gamma_5\,\varepsilon^{\mu\nu\rho\sigma}\gamma^\sigma\,, \tag{C.28}$$

$$\tfrac{1}{24}\,[\gamma^\mu,\gamma^\nu,\gamma^\alpha,\gamma^\beta] = -\gamma_5\,\varepsilon^{\mu\nu\alpha\beta}\,. \tag{C.29}$$

The various contractions of $\varepsilon-$tensors are given by

$$\begin{aligned}
\varepsilon^{\mu\nu\rho\lambda}\,\varepsilon^{\alpha\beta\gamma\lambda} &= \delta^{\mu\alpha}\,(\delta^{\nu\beta}\,\delta^{\rho\gamma} - \delta^{\nu\gamma}\,\delta^{\rho\beta}) + \delta^{\mu\beta}\,(\delta^{\nu\gamma}\,\delta^{\rho\alpha} - \delta^{\nu\alpha}\,\delta^{\rho\gamma}) \\
&\quad + \delta^{\mu\gamma}\,(\delta^{\rho\beta}\,\delta^{\nu\alpha} - \delta^{\rho\alpha}\,\delta^{\nu\beta})\,, \\
\tfrac{1}{2}\,\varepsilon^{\mu\nu\lambda\sigma}\,\varepsilon^{\alpha\beta\lambda\sigma} &= \delta^{\mu\alpha}\,\delta^{\nu\beta} - \delta^{\mu\beta}\,\delta^{\nu\alpha}\,, \\
\tfrac{1}{6}\,\varepsilon^{\mu\lambda\sigma\tau}\,\varepsilon^{\alpha\lambda\sigma\tau} &= \delta^{\mu\alpha}\,, \\
\tfrac{1}{24}\,\varepsilon^{\lambda\sigma\tau\omega}\,\varepsilon^{\lambda\sigma\tau\omega} &= 1\,.
\end{aligned} \tag{C.30}$$

The $\varepsilon-$tensor satisfies $a^{\{\mu}\varepsilon^{\alpha\beta\gamma\delta\}} = 0$, where a^μ is an arbitrary four-vector and $\{\dots\}$ denotes a symmetrization of indices.

Momentum integrations. Four-momenta are conveniently expressed through hyperspherical coordinates:

$$p^\mu = \sqrt{p^2}\begin{bmatrix} \sqrt{1-z^2}\,\sqrt{1-y^2}\,\sin\phi \\ \sqrt{1-z^2}\,\sqrt{1-y^2}\,\cos\phi \\ \sqrt{1-z^2}\,\,y \\ z \end{bmatrix}. \tag{C.31}$$

For a particle in its rest frame, this corresponds to $p^\mu = (\mathbf{0}, im)$. (Actually, in Euclidean space it does not matter *where* we put the mass since each direction is treated equally.) A four-momentum integration reads

$$\int\frac{d^4p}{(2\pi)^4} = \frac{1}{(2\pi)^4}\,\frac{1}{2}\int_0^\infty dp^2\,p^2\int_{-1}^1 dz\,\sqrt{1-z^2}\int_{-1}^1 dy\int_0^{2\pi} d\phi\,, \tag{C.32}$$

where $\frac{1}{2}\,dp^2\,p^2 = dp\,p^3$ and

$$\int d\Omega_4 = \int_{-1}^1 dz\,\sqrt{1-z^2}\int_{-1}^1 dy\int_0^{2\pi} d\phi = 2\pi^2 \tag{C.33}$$

is the integral over the unit sphere in four dimensions.

Spinors. The positive- and negative-energy onshell spinors for spin-1/2 particles satisfy the Dirac equations

$$\begin{aligned} (i\not{p}+m)\,u(\boldsymbol{p}) &= 0 = \bar{u}(\boldsymbol{p})\,(i\not{p}+m)\,, \\ (i\not{p}-m)\,v(\boldsymbol{p}) &= 0 = \bar{v}(\boldsymbol{p})\,(i\not{p}-m)\,, \end{aligned} \tag{C.34}$$

where the conjugate spinor is $\bar{u}(\boldsymbol{p}) = u(\boldsymbol{p})^\dagger \gamma^4$. Since the onshell spinors only depend on $\boldsymbol{p}$ they are the same as in Minkowski space; for example in the standard representation:

$$u_s(\boldsymbol{p}) = \sqrt{\frac{E_p+m}{2m}} \begin{pmatrix} \xi_s \\ \frac{\boldsymbol{p}\cdot\boldsymbol{\tau}}{E_p+m}\,\xi_s \end{pmatrix} \tag{C.35}$$

with

$$\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad E_p = \sqrt{\boldsymbol{p}^2+m^2}\,.$$

We have normalized them to unity,

$$\begin{aligned} \bar{u}_s(\boldsymbol{p})\,u_{s'}(\boldsymbol{p}) &= -\bar{v}_s(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) = \delta_{ss'}\,, \\ \bar{u}_s(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) &= \bar{v}_s(\boldsymbol{p})\,u_{s'}(\boldsymbol{p}) = 0\,, \end{aligned} \tag{C.36}$$

and their completeness relations define the positive- and negative-energy projectors:

$$\begin{aligned} \sum_s u_s(\boldsymbol{p})\,\bar{u}_s(\boldsymbol{p}) &= \frac{-i\not{p}+m}{2m} = \Lambda_+(p)\,, \\ \sum_s v_s(\boldsymbol{p})\,\bar{v}_s(\boldsymbol{p}) &= \frac{-i\not{p}-m}{2m} = -\Lambda_-(p)\,. \end{aligned} \tag{C.37}$$

Therefore, $\Lambda_+(p)\,u(\boldsymbol{p}) = u(\boldsymbol{p})$ and $\Lambda_-(p)\,u(\boldsymbol{p}) = 0$.