## 4.4 Chiral effective field theories

In the discussion so far we have seen that the information on hadrons that can be easily and directly extracted from the QCD Lagrangian is limited: some exact statements are possible, but in practice one needs numerical calculations and/or models to describe the dynamics of the theory. On the other hand, analytic calculations are still possible if we exploit the symmetries of QCD. In particular, the near chiral symmetry of the QCD Lagrangian and its spontaneous breaking can be used to construct low-energy effective theories of QCD, which are not formulated in terms of quarks and gluons but rather with hadrons as effective degrees of freedom. The fact that the pion mass is so much smaller than all other hadronic energy scales makes a perturbative expansion in powers of momenta and pion masses possible. The resulting field theory is called **chiral perturbation theory (ChPT)** and allows one to make rigorous statements as long as the momenta and pion masses are small.

### 4.4.1 Sigma model

**Linear sigma model.** We start with the **linear sigma model**, which is the prototype of an effective field theory that implements spontaneous chiral symmetry breaking. In its basic version it describes the interaction of nucleons with pions and scalar mesons:

- The nucleon is represented by spinor fields $\psi(x)$, $\overline{\psi}(x)$ which are isospin doublets, i.e., they transform under the fundamental representation of $SU(2)_f$.
- The three pions correspond to an isospin triplet $\pi_a(x)$ of pseudoscalar fields.
- The scalar meson $\sigma(x)$ is an isoscalar and identified with the $\sigma/f_0(500)$.

One could extend this by including more meson fields such as the $\rho$ meson or other baryon fields, and various **quark-meson models** have been constructed by interpreting the spinors not as nucleons but as quarks.

We combine the pions and the scalar meson into a **meson matrix** $\phi$, which is a matrix in Dirac and flavor space and depends linearly on $\pi_a$ and $\sigma$:

$$\phi = \sigma + i\gamma_5\, \boldsymbol{\tau}\cdot\boldsymbol{\pi}\,. \tag{4.4.1}$$

Here, $\tau_a$ are the Pauli matrices which are related to the $SU(2)_f$ generators by $\mathsf{t}_a = \tau_a/2$. We defined the 'length' of $\phi$ that will enter in the mass term by

$$|\phi|^2 := \tfrac{1}{2}\,\mathrm{Tr}\left\{\phi^\dagger\phi\right\} = \tfrac{1}{2}\,\mathrm{Tr}\left\{(\sigma - i\gamma_5\,\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\sigma + i\gamma_5\,\boldsymbol{\tau}\cdot\boldsymbol{\pi})\right\} = \sigma^2 + \boldsymbol{\pi}^2\,, \tag{4.4.2}$$

where we used the identities (same indices are summed over)

$$(\boldsymbol{\tau}\cdot\boldsymbol{\pi})^2 = \pi_a\,\pi_b\,\Big(\underbrace{\tfrac{1}{2}[\tau_a,\tau_b]}_{if_{abc}\,\tau_c} + \underbrace{\tfrac{1}{2}\{\tau_a,\tau_b\}}_{\delta_{ab}}\Big) = \boldsymbol{\pi}^2\,, \qquad \mathrm{Tr}\left\{\tau_a\,\tau_b\right\} = 2\delta_{ab} \tag{4.4.3}$$

with $f_{abc} = \varepsilon_{abc}$ in $SU(2)$. Likewise, for the kinetic term we have

$$|\partial_\mu\phi|^2 = \tfrac{1}{2}\,\mathrm{Tr}\left\{\partial_\mu\phi^\dagger\,\partial^\mu\phi\right\} = (\partial_\mu\,\sigma)^2 + (\partial_\mu\,\boldsymbol{\pi})^2\,. \tag{4.4.4}$$

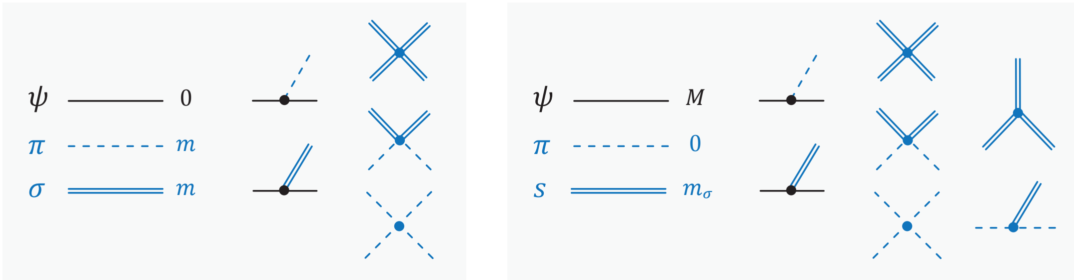


FIG. 4.10: Field content of the sigma model before (left) and after (right) spontaneous chiral symmetry breaking.

The Lagrangian then reads as follows:

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - g\,\phi)\,\psi + \tfrac{1}{2}\left(|\partial_\mu\phi|^2 - m^2|\phi|^2\right) - V(|\phi|^2)\,, \tag{4.4.5}$$

where the meson matrix couples to the spinors through a Yukawa interaction $\bar{\psi}\,\phi\,\psi$. We have assigned the same mass $m$ to each meson, whereas the nucleon at this point is massless. The potential $V$ depends on powers of the meson matrix and we will specify it below. For four-point interactions, the field content of this theory is shown in the left panel of Fig. 4.10.

Let us impose **chiral symmetry** $SU(2)_V \times SU(2)_A$ on the Lagrangian, where the fermions transform under Eqs. (3.1.18–3.1.19):

$$V: \quad U = \exp\left(i\varepsilon_a\,\frac{\tau_a}{2}\right) \quad \Rightarrow \quad U' = U\psi\,, \quad \bar{\psi}' = \bar{\psi}\,U^\dagger\,, \tag{4.4.6}$$

$$A: \quad U = \exp\left(i\gamma_5\,\varepsilon_a\,\frac{\tau_a}{2}\right) \quad \Rightarrow \quad U' = U\psi\,, \quad \bar{\psi}' = \bar{\psi}\,U\,. \tag{4.4.7}$$

The fermion kinetic term is invariant under both operations, i.e., chirally symmetric:

$$(\bar{\psi}\,i\partial\!\!\!/\,\psi)' = \left\{\begin{array}{ll} \bar{\psi}\,U^\dagger i\partial\!\!\!/\,U\psi & \ldots V \\ \bar{\psi}\,U i\partial\!\!\!/\,U\psi & \ldots A \end{array}\right\} = \bar{\psi}\,i\partial\!\!\!/\,U^\dagger U\psi = \bar{\psi}\,i\partial\!\!\!/\,\psi\,. \tag{4.4.8}$$

We have not yet defined how the meson fields transform under chiral symmetry. To do so, we impose invariance of the meson-fermion coupling term $\bar{\psi}\,\phi\,\psi$:

$$V: \quad (\bar{\psi}\,\phi\,\psi)' = \bar{\psi}\,U^\dagger\phi'\,U\psi \overset{!}{=} \bar{\psi}\,\phi\,\psi \quad \Rightarrow \quad \phi' = U\phi\,U^\dagger\,, \tag{4.4.9}$$

$$A: \quad (\bar{\psi}\,\phi\,\psi)' = \bar{\psi}\,U\phi'\,U\psi \overset{!}{=} \bar{\psi}\,\phi\,\psi \quad \Rightarrow \quad \phi' = U^\dagger\phi\,U^\dagger\,. \tag{4.4.10}$$

The infinitesimal transformations for the $\pi_a$ and $\sigma$ fields then become

$$V: \quad \sigma' = \sigma\,, \qquad \pi'_a = \pi_a - f_{abc}\,\varepsilon_b\,\pi_c\,, \tag{4.4.11}$$

$$A: \quad \sigma' = \sigma + \varepsilon_a\,\pi_a\,, \qquad \pi'_a = \pi_a - \varepsilon_a\,\sigma\,. \tag{4.4.12}$$

Observe that the $SU(2)_A$ transformation mixes the $\sigma$ with the pion fields! This is why they belong together and we needed *both* of them in constructing a chirally invariant Lagrangian.

From the transformation behavior of the meson matrix $\phi$ it is clear that the remaining terms in the Lagrangian $|\phi|^2 = \frac{1}{2}\,\mathrm{Tr}\left\{\phi^\dagger\phi\right\}$, $|\partial_\mu\phi|^2$ and $V(|\phi|^2)$ are also chirally invariant. For the individual fields this entails

$$V: \quad \sigma'^2 = \sigma^2\,, \qquad\qquad \boldsymbol{\pi}'^2 = \boldsymbol{\pi}^2 + 2f_{abc}\,\pi_a\,\pi_b\,\varepsilon_c = \boldsymbol{\pi}^2\,, \tag{4.4.13}$$

$$A: \quad \sigma'^2 = \sigma^2 + 2\sigma\pi_a\,\varepsilon_a\,, \qquad \boldsymbol{\pi}'^2 = \boldsymbol{\pi}^2 - 2\sigma\pi_a\,\varepsilon_a\,. \tag{4.4.14}$$

While $SU(2)_V$ leaves both $\sigma^2$ and $\boldsymbol{\pi}^2$ invariant, $SU(2)_A$ only preserves their combination $\sigma^2 + \boldsymbol{\pi}^2$. Moreover, renormalizability entails that the possible self-interactions in the potential $V(|\phi|^2)$ can be of order four at most, since the couplings for higher interactions would have a negative mass dimension. A $|\phi|^4$ interaction then leads to the quartic interaction vertices shown in Fig. 4.10.

In this initial Lagrangian, chiral symmetry demands that both mesons must have the same mass $m$ and coupling strength $g$. Recalling Eq. (3.1.49), we deliberately did not include a mass term for the nucleon since it would break chiral symmetry. Below we will generate a nucleon mass and eliminate the pion mass by means of spontaneous chiral symmetry breaking.

The vector and axialvector **currents** corresponding to the $SU(2)_V$ and $SU(2)_A$ symmetries can be derived from their definition in (3.1.2). They pick up additional terms from the meson fields $\sigma$ and $\pi_a$:

$$V_a^\mu = \bar\psi\,\gamma^\mu\,\mathsf{t}_a\,\psi + f_{abc}\,\pi_b\,\partial^\mu\pi_c\,, \qquad A_a^\mu = \bar\psi\,\gamma^\mu\gamma_5\,\mathsf{t}_a\,\psi + \sigma\,\partial^\mu\pi_a - \pi_a\,\partial^\mu\sigma\,. \tag{4.4.15}$$

These currents are conserved because the Lagrangian is chirally invariant. The classical **equations of motion** of the linear sigma model are

$$\begin{aligned} \partial\!\!\!/\,\psi &= -ig\,\phi\,\psi\,, & (\Box + m^2)\,\sigma &= -g\,\bar\psi\,\psi\,, \\ \bar\psi\,\overleftarrow{\partial\!\!\!/} &= ig\,\bar\psi\,\phi\,, & (\Box + m^2)\,\pi_a &= -2ig\,\bar\psi\,\gamma_5\,\mathsf{t}_a\,\psi\,, \end{aligned} \tag{4.4.16}$$

up to terms coming from the potential $V(|\phi|^2)$.

We note that one could rewrite the linear sigma model in terms of a meson matrix $\Sigma$ which is a matrix in flavor space only:

$$\Sigma := \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\,. \tag{4.4.17}$$

Employing the chiral projectors $\mathsf{P}_\pm = (\mathbb{1} \pm \gamma_5)/2$ from Eq. (3.1.42), we have

$$\begin{aligned} \phi = \sigma + i\gamma_5\,\boldsymbol{\tau}\cdot\boldsymbol{\pi} &= (\mathsf{P}_+ + \mathsf{P}_-)\,\sigma + (\mathsf{P}_+ - \mathsf{P}_-)\,i\boldsymbol{\tau}\cdot\boldsymbol{\pi} \\ &= \mathsf{P}_+\,(\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}) + \mathsf{P}_-\,(\sigma - i\boldsymbol{\tau}\cdot\boldsymbol{\pi}) \\ &= \mathsf{P}_+\Sigma + \mathsf{P}_-\Sigma^\dagger = \mathsf{P}_+\Sigma\,\mathsf{P}_+ + \mathsf{P}_-\Sigma^\dagger\,\mathsf{P}_-\,. \end{aligned} \tag{4.4.18}$$

With the definition (3.1.43) of the right- and left-handed spinors, $\psi_\omega = \mathsf{P}_\omega\,\psi$ and $\bar\psi_\omega = \bar\psi\,\mathsf{P}_{-\omega}$, the Yukawa coupling becomes

$$\bar\psi\,\phi\,\psi = \bar\psi_-\Sigma\,\psi_+ + \bar\psi_+\Sigma^\dagger\psi_-\,. \tag{4.4.19}$$

The remaining terms, defined via (4.4.2), have the same form as before:

$$|\phi|^2 = |\Sigma|^2\,, \qquad |\partial_\mu\phi|^2 = |\partial_\mu\Sigma|^2\,. \tag{4.4.20}$$

With the transformation of the chiral spinors in Eq. (3.1.46), $\psi'_\omega = U_\omega\,\psi_\omega$ and $\bar\psi'_\omega = \bar\psi_\omega\,U_\omega^\dagger$, chiral symmetry demands

$$\bar\psi'_-\,\Sigma'\,\psi'_+ + \bar\psi'_+\,\Sigma'^\dagger\psi'_- = \bar\psi_-\,U_-^\dagger\Sigma'\,U_+\psi_+ + \bar\psi_+\,U_+^\dagger\Sigma'^\dagger\,U_-\psi_- \stackrel{!}{=} \bar\psi_-\Sigma\,\psi_+ + \bar\psi_+\Sigma^\dagger\psi_-\,, \tag{4.4.21}$$

hence the matrix $\Sigma$ must transforms under $SU(2)_L \times SU(2)_R$ as $\Sigma' = U_-\Sigma\,U_+^\dagger$.

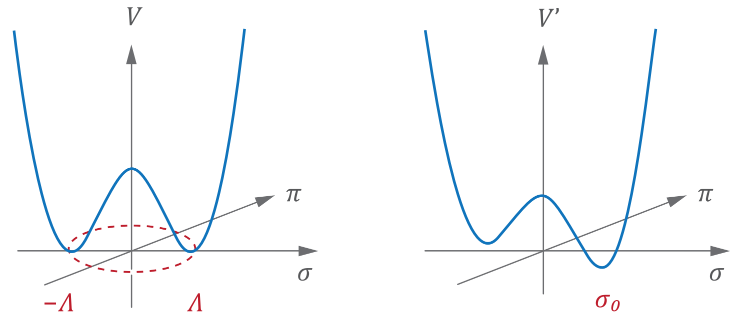


FIG. 4.11: Mexican hat potential of Eq. (4.4.22), with minima along the chiral circle. The right figure includes the explicit symmetry-breaking term of Eq. (4.4.29).

**Spontaneous chiral symmetry breaking.** Next, we want to generate a mass for the fermions and also get rid of the pion mass. To this end we drop the identification of $m$ with the masses of $\sigma$ and $\boldsymbol{\pi}$. Instead we interpret it as a scale $\Lambda$ via $-m^2 =: \lambda\Lambda^2$ that we absorb into the potential:

$$V(|\phi|^2) = \frac{\lambda}{4}\,|\phi|^4 - \frac{\lambda\Lambda^2}{2}\,|\phi|^2 = \frac{\lambda}{4}\left(|\phi|^2 - \Lambda^2\right)^2 - \frac{\lambda}{4}\,\Lambda^4\,. \tag{4.4.22}$$

Constant terms can always be dropped from the Lagrangian. The remainder is the **mexican hat** potential shown in Fig. 4.11, which has minima along the 'chiral circle' $|\phi|^2 = \sigma^2 + \boldsymbol{\pi}^2 = \Lambda^2$. Note that this is still a chirally symmetric condition and the Lagrangian is invariant under chiral symmetry as before.

However, in this way we have prepared the groundwork that triggers a spontaneous symmetry breaking (SSB) in the quantum field theory. Recall the discussion of the quantum effective action $\Gamma[\varphi]$ and the classical field $\varphi(x) = \langle\phi(x)\rangle_J$ around Eq. (2.2.42). If we set the sources $J = 0$, then also $\varphi(x) = 0$. The first derivative of $\Gamma[\varphi]$ vanishes and the higher derivatives are the 1PI correlation functions for the field $\phi(x)$:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}\bigg|_{\varphi=0} = 0\,, \qquad \frac{\delta^n\Gamma[\varphi]}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\bigg|_{\varphi=0} = \Gamma^{(n)}_\phi(x_1,\dots x_n)\,. \tag{4.4.23}$$

In the presence of a non-zero vacuum expectation value, setting $J = 0$ entails $\varphi(x) = v$ and these relations are modified as follows:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}\bigg|_{\varphi=v} = 0\,, \qquad \frac{\delta^n\Gamma[\varphi]}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\bigg|_{\varphi=v} = \Gamma^{(n)}_{\phi-v}(x_1,\dots x_n)\,. \tag{4.4.24}$$

The higher derivatives are the 1PI correlation functions for the field $\phi(x) - v$ and the 'one-point function' still vanishes for $\varphi(x) = v$, which therefore extremizes the effective action. Since the classical potential gives the tree-level contribution to $\Gamma[\varphi]$, its minimum determines the leading-order result for the VEV.

Then again, the minimum of the mexican hat is still a chirally symmetric condition. What actually breaks the chiral symmetry of the vacuum is parity invariance, which entails $\langle 0|\pi_a|0\rangle = 0$ and leaves only $\sigma_0 = \langle 0|\sigma|0\rangle = \pm\Lambda$, i.e., it singles out two points on the chiral circle. To determine the true ground state, one must introduce an explicit symmetry-breaking term that tilts the potential towards one absolute minimum.

The next step is to expand the $\sigma$ field around its minimum by introducing a new fluctuating field $s$. There are different ways to do so; one possible choice is $\sigma = \Lambda + s$. Since $\Lambda$ is a constant, we have $\partial_\mu \sigma = \partial_\mu s$ and the form of the kinetic term for the mesons remains unchanged:

$$\frac{1}{2}\,|\partial_\mu \phi|^2 = \frac{1}{2}\left((\partial_\mu \sigma)^2 + (\partial_\mu \boldsymbol{\pi})^2\right) \cong -\frac{1}{2}\,(s\,\Box\,s + \boldsymbol{\pi}\,\Box\,\boldsymbol{\pi})\,. \tag{4.4.25}$$

Instead, the potential becomes

$$\begin{aligned} V(|\phi|^2) = \frac{\lambda}{4}\left(|\phi|^2 - \Lambda^2\right)^2 &= \frac{\lambda}{4}\left((\Lambda+s)^2 + \boldsymbol{\pi}^2 - \Lambda^2\right)^2 = \frac{\lambda}{4}\left(s^2 + \boldsymbol{\pi}^2 + 2\Lambda s\right)^2 \\ &= \lambda\left[\frac{1}{4}\,(s^2+\boldsymbol{\pi}^2)^2 + \Lambda s\,(s^2+\boldsymbol{\pi}^2) + \Lambda^2\,s^2\right]. \end{aligned} \tag{4.4.26}$$

Expressed in terms of $s$ and $\boldsymbol{\pi}$, the Lagrangian (4.4.5) reads explicitly:

$$\begin{aligned} \mathcal{L} &= \overline{\psi}\,(i\partial\!\!\!/ - g\Lambda)\,\psi - g\overline{\psi}\,(s + i\gamma_5\,\boldsymbol{\tau}\cdot\boldsymbol{\pi})\,\psi \\ &\quad - \frac{1}{2}\,s\,(\Box + 2\lambda\Lambda^2)\,s - \frac{1}{2}\,\boldsymbol{\pi}\,\Box\,\boldsymbol{\pi} - \lambda\,\Lambda\,(s^3 + s\boldsymbol{\pi}^2) - \frac{\lambda}{4}\,(s^4 + 2s^2\boldsymbol{\pi}^2 + \boldsymbol{\pi}^4)\,. \end{aligned} \tag{4.4.27}$$

In this way we have generated a nucleon mass $M$, a scalar mass $m_\sigma$, and two new cubic interaction vertices $\sim s^3$ and $\sim s\pi^2$ (see right panel of Fig. 4.10):

$$M = g\,\Lambda, \qquad m_\sigma = \sqrt{2\lambda}\,\Lambda, \qquad g_{sss} = g_{\pi\pi s} = \lambda\,\Lambda. \tag{4.4.28}$$

The pions remain massless, hence they are the three Goldstone bosons of the spontaneously broken $SU(2)_A$. Observe that since we only redefined the fields, the Lagrangian is still the same as before and therefore chirally invariant (despite the fermion mass term!). The symmetry is merely 'hidden'. However, the ground state is not invariant and thus chiral symmetry is spontaneously broken in the QFT.

**Explicit chiral symmetry breaking.** Since the pions in nature have a mass, we can add a term to the Lagrangian that breaks chiral symmetry explicitly,

$$V' = V - m_\pi^2\,\Lambda\,\sigma \quad \Leftrightarrow \quad \mathcal{L}' = \mathcal{L} + m_\pi^2\,\Lambda\,\sigma\,, \tag{4.4.29}$$

where we already named the coefficient accordingly. The potential is now tilted, and the absolute minimum appears at

$$\left.\frac{\partial V'}{\partial \sigma}\right|_{\substack{\sigma=\sigma_0,\\ \pi_a=0}} \stackrel{!}{=} 0 \quad \Rightarrow \quad \lambda\,(\sigma_0^2 - \Lambda^2) = m_\pi^2\,\frac{\Lambda}{\sigma_0} \approx m_\pi^2 \quad \Rightarrow \quad \sigma_0 = +\sqrt{\Lambda^2 + \frac{m_\pi^2}{\lambda}}\,. \tag{4.4.30}$$

If we expand around the new minimum and insert $\sigma = \sigma_0 + s$ into the potential (4.4.22), we generate a mass term $\sim -\frac{1}{2}\,m_\pi^2\,\boldsymbol{\pi}^2$ for the pion. The remaining Lagrangian has the same form as in Eq. (4.4.27) at first order in $m_\pi^2$, but instead of the relations (4.4.28) we find:

$$M = g\,\sqrt{\Lambda^2 + \frac{m_\pi^2}{\lambda}}\,, \quad m_\sigma = \sqrt{2\lambda}\,\sqrt{\Lambda^2 + \frac{3m_\pi^2}{2\lambda}}\,, \quad g_{sss} = g_{\pi\pi s} = \lambda\,\sqrt{\Lambda^2 + \frac{m_\pi^2}{\lambda}}\,. \tag{4.4.31}$$

The resulting evolution of the nucleon mass with $m_\pi^2$ already resembles the outcome of realistic calculations in QCD, which we sketched earlier in Fig. 4.7. In the linear sigma model, the nucleon mass in the chiral limit is $g\Lambda$.

**Non-linear representations.** The linear sigma model needs both pions and a scalar field to respect chiral symmetry. This is not very satisfactory because the actual $\sigma/f_0(500)$ is a broad resonance and it seems unnatural that it would play the fundamental role suggested by the linear sigma model. While we cannot simply set $s=0$ in the way we introduced it above ($\sigma = \Lambda + s$) without breaking the chiral symmetry of the Lagrangian, we can eliminate the $\sigma$ meson by allowing the meson matrix to be **nonlinear** in the pion fields.

Let us introduce a new scalar field $s$ and new pion fields $\varphi_a$ by

$$\phi = (\Lambda + s)\,\Omega\,, \quad \Omega = \exp\left(i\gamma_5\,\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,\frac{\alpha(z)}{\Lambda z}\right) = \cos\alpha(z) + i\gamma_5\,\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,\frac{\sin\alpha(z)}{\Lambda z}\,. \quad (4.4.32)$$

Here we defined

$$(\boldsymbol{\tau}\cdot\boldsymbol{\varphi})^2 = \boldsymbol{\varphi}^2 = \Lambda^2 z^2\,, \quad (4.4.33)$$

where $z$ is the dimensionless 'length' of the pion field, $\alpha(z)$ is some function of $z$, and we used $e^{iA\alpha} = \cos\alpha + iA\,\sin\alpha$ for $A^2=1$. As a consequence, the original fields $\sigma$ and $\pi_a$ are related to the new ones by

$$\sigma = (\Lambda+s)\,\cos\alpha(z)\,, \qquad \pi_a = (\Lambda+s)\,\frac{\varphi_a}{\Lambda z}\,\sin\alpha(z)\,. \quad (4.4.34)$$

The advantage of doing this is that $\Omega$ depends only on the new pion fields (but on *all* powers of them), and because of $|\phi|^2 = (\Lambda+s)^2$ the potential depends only on the scalar field:

$$V(|\phi|^2) = \frac{\lambda}{4}\left(|\phi|^2 - \Lambda^2\right)^2 = \lambda\left(\frac{s^4}{4} + \Lambda s^3 + \Lambda^2 s^2\right). \quad (4.4.35)$$

Because $|\phi|^2$ is chirally symmetric, in this way we have achieved a chirally symmetric separation of the scalar and pion fields. In turn, the kinetic term for the mesons becomes more complicated and also encodes the pion's self-interactions via derivative couplings. With $\partial_\mu\phi = \partial_\mu s\,\Omega + (\Lambda+s)\,\partial_\mu\Omega$ we find

$$\begin{aligned}
|\partial_\mu\phi|^2 &= \tfrac{1}{2}\,\mathrm{Tr}\left\{\partial_\mu\phi^\dagger\,\partial^\mu\phi\right\} \\
&= \tfrac{1}{2}\,\mathrm{Tr}\Big\{\partial_\mu s\,\partial^\mu s + (\Lambda+s)\,\partial^\mu s\,\underbrace{\left(\partial_\mu\Omega^\dagger\,\Omega + \Omega^\dagger\,\partial_\mu\Omega\right)}_{\partial_\mu\,(\Omega^\dagger\Omega)=0} + (\Lambda+s)^2\,\partial_\mu\Omega^\dagger\,\partial^\mu\Omega\Big\} \\
&= (\partial_\mu s)^2 + (\Lambda+s)^2\,|\partial_\mu\Omega|^2\,,
\end{aligned}$$

where $|\partial_\mu\Omega|^2$ is a complicated function of the pion fields. The explicit calculation yields

$$|\partial_\mu\Omega|^2 = \frac{1}{z^2}\left[\frac{(\partial_\mu\,\boldsymbol{\varphi})^2}{\Lambda^2}\,\sin^2\alpha + \frac{(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})^2}{\Lambda^4}\left(\alpha'(z)^2 - \frac{\sin^2\alpha}{z^2}\right)\right]. \quad (4.4.36)$$

Depending on the function $\alpha(z)$, we could work with the

- **exponential representation:** $\alpha(z) = z \;\Rightarrow\; \Omega = \exp\left(i\gamma_5\,\frac{\boldsymbol{\tau}\cdot\boldsymbol{\varphi}}{\Lambda}\right)$,
- **square-root representation:** $\sin\alpha(z) = z \;\Rightarrow\; \Omega = \sqrt{1-z^2} + i\gamma_5\,\frac{\boldsymbol{\tau}\cdot\boldsymbol{\varphi}}{\Lambda}$.

In any case, the Lagrangian in terms of the new fields becomes

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - g\,(\Lambda + s)\,\Omega\right)\psi - \frac{1}{2}\,s\,(\Box + 2\lambda\,\Lambda^2)\,s - \lambda\left(\frac{s^4}{4} + \Lambda s^3\right) + \frac{1}{2}\,(\Lambda + s)^2\,|\partial_\mu\Omega|^2.$$

Diagrammatically it still contains the Feynman rules from the right panel of Fig. 4.10, with the identifications in Eq. (4.4.28), but due to the appearance of $\Omega$ there are new vertices with pion legs: in fact, once we expand the exponential, there are infinitely many of them! While this looks very different from the Lagrangians (4.4.5) or (4.4.27), in principle it is still the same theory since all we have done is renaming the fields. Indeed one can show that onshell scattering amplitudes obtained from either of these representations are identical.

**Non-linear sigma model.** The main advantage of arranging the fields in this way is the following: because we separated the fields $s$ and $\varphi_a$ in a chirally invariant manner, setting $s = 0$ does no longer break chiral symmetry and we can safely eliminate it from the theory. The resulting Lagrangian is

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_\pi = \overline{\psi}\left(i\partial\!\!\!/ - g\Lambda\,\Omega\right)\psi + \frac{\Lambda^2}{2}\,|\partial_\mu\Omega|^2 \tag{4.4.37}$$

and contains nucleons and pions only. The pionic part $\mathcal{L}_\pi$, where the self-interactions of the pions enter via derivative couplings, is called **nonlinear sigma model**. Note that we could have also obtained it by setting $\sigma^2 + \boldsymbol{\pi}^2 = \Lambda^2$ from the beginning, i.e., by restricting the fields to the chiral circle and thereby eliminating the $\sigma$ field as an independent degree of freedom.

Unfortunately, the chirally invariant separation of scalar and pion fields comes at a price, namely the loss of renormalizability. The reason is that $\Omega$ and $|\partial_\mu\Omega|^2$ contain all powers of the pion field $\varphi_a$. Suppose we work with the exponential representation $\alpha(z) = z$:

$$|\partial_\mu\Omega|^2 = \frac{1}{z^2}\left[\frac{(\partial_\mu\,\boldsymbol{\varphi})^2}{\Lambda^2}\,\sin^2 z + \frac{(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})^2}{\Lambda^4}\left(1 - \frac{\sin^2 z}{z^2}\right)\right]. \tag{4.4.38}$$

The Taylor expansion of $(\sin z/z)^2$ gives

$$\frac{\sin^2 z}{z^2} = 1 - \frac{z^2}{3} + \frac{2z^4}{45} + \dots = 1 - z^2\sum_{r=0}^{\infty} c_r\,z^{2r}\,, \tag{4.4.39}$$

so that $\mathcal{L}_\pi$ becomes

$$\mathcal{L}_\pi = \frac{1}{2}\,(\partial_\mu\,\boldsymbol{\varphi})^2 + \frac{1}{2}\left((\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi})^2 - \boldsymbol{\varphi}^2\,(\partial_\mu\,\boldsymbol{\varphi})^2\right)\sum_{r=0}^{\infty}\frac{c_r}{(\Lambda^2)^{r+1}}\,(\boldsymbol{\varphi}^2)^r\,. \tag{4.4.40}$$

The first term is the inverse tree-level propagator. The second contains an infinite number of tree-level vertices with even numbers of pion legs and derivative couplings (Fig. 4.12): The term with $r = 0$ returns a four-point vertex with coupling constant $c_0/\Lambda^2$, the term with $r = 1$ a six-point vertex with coupling constant $c_1/\Lambda^4$, and so on. As a result, the perturbative expansion of an $n$-point function not only contains infinitely many loop diagrams but also depends on infinitely many *vertices*.

Fig. 4.12: Tree-level diagrams contained in the Lagrangians (4.4.37) and (4.4.64).

Even worse, the couplings carry negative mass dimensions and therefore these interactions are non-renormalizable. Because we deleted the field $s$, the non-linear sigma model is no longer equivalent to the original Lagrangian, which was renormalizable even though in the end this was no longer obvious (still, we could have transformed back to the original fields $\sigma$ and $\pi_a$). In practice this means that each $n$-point function produces new divergences, so we would also need infinitely many renormalization conditions and the theory loses its predictivity.

While this would make practical applications hopeless, the fact that all couplings contain **derivatives** opens up a new interpretation: derivatives become momenta in momentum space, and higher powers of momenta are suppressed at low energies. If we can show that higher-loop diagrams also correlate with higher momentum powers, then we can stop the expansion at some given order and fix the necessary renormalization constants at that order by outside information, e.g., from experimental data. The convergence radius is then limited to low momenta and low energies; hence, we can interpret the model as a **low-energy effective theory**. In fact, the nonlinear sigma model $\mathcal{L}_\pi$ constitutes the lowest-order term in chiral perturbation theory.

**Weinberg's power counting.** For illustration, let us go back to a $\phi^p$ theory, where we add up the $\phi^p$ couplings in the Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \lambda_4\,\phi^4 + \frac{\lambda_6}{\Lambda^2}\,\phi^6 + \frac{\lambda_8}{\Lambda^4}\,\phi^8 \dots \qquad (4.4.41)$$

Because the non-renormalizable couplings for $p > 4$ carry negative mass dimensions, we pulled out powers of a scale $\Lambda$ so that the $\lambda_p$ are dimensionless. In this case the formula (2.3.62) which we established earlier generalizes to

$$[\Gamma_n] = 4L - 2I + \sum_p (4-p)\,V_p\,, \qquad \sum_p V_p = V\,. \qquad (4.4.42)$$

Here, $[\Gamma_n]$ is the mass dimension of a given $n$-point function. For some perturbative loop diagram that contributes to $\Gamma_n$, $L$ is the number of loops, $I$ the number of propagators and $V_p$ the number of $\phi^p$ vertices, with $V$ the total number of vertices in the diagram. $D = 4L - 2I$ is the degree of divergence of the diagram.

Now observe that in order to preserve the mass dimension of the $n$-point function, internal momentum powers in a loop must translate to powers of the external momenta and masses. This is easiest to see in dimensional regularization from the loop formulas (2.3.40–2.3.41):

$$I^{(4)}_{nm} = \int \frac{d^4 l_E}{(2\pi)^4}\,\frac{(l_E^2)^m}{(l_E^2+\Delta)^n} \propto \Delta^{2+m-n}\,, \qquad (4.4.43)$$

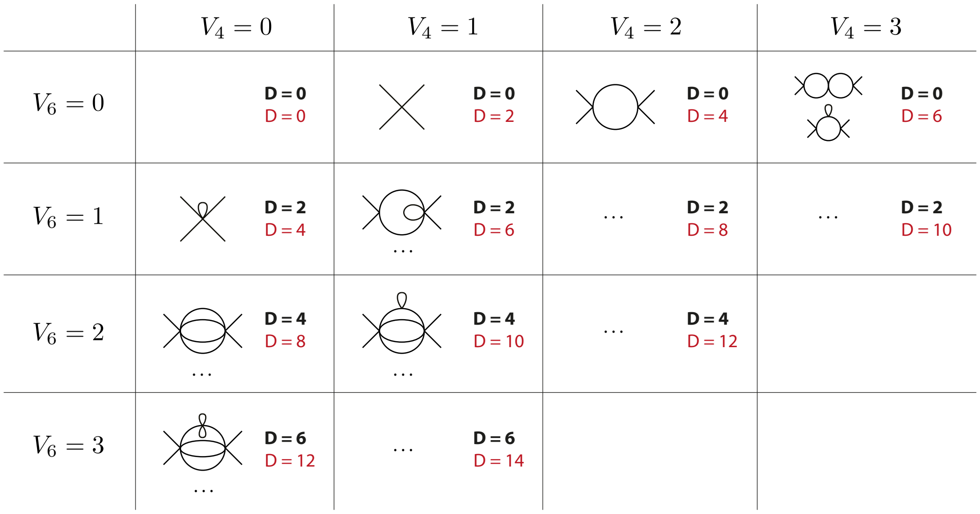


FIG. 4.13: Loop diagrams contributing to a 1PI four-point function with four- and six-point interactions. The degree of divergence $D$ for ordinary couplings with $d_\chi = 0$ is given in bold black font and the one for derivative couplings with $d_\chi = 2$ in red, cf. Eq. (4.4.48).

where the quantity $\Delta$ defined in Eq. (2.3.32) has mass dimension two and depends on the external momenta and the masses in the loop. It is attached to an expression that contains divergent $1/\varepsilon$ terms and finite parts, where the former drop out after renormalization. The renormalization scale $M$ (or $\mu$) only enters through logarithms. Likewise, had we employed a cutoff regulator, the divergent terms would scale with powers of the cutoff and drop out after renormalization, whereas the finite pieces scale with powers of the external momenta and the masses.

If a diagram contains higher $\phi^p$ vertices, its degree of divergence $D$ raises according to Eq. (4.4.42):

$$D = [\Gamma_n] + \sum_p (p-4)\, V_p \,. \tag{4.4.44}$$

The reason is that those vertices come with higher powers of $\Lambda^2$ in the denominators, which must be compensated by momentum powers in the numerators to preserve the mass dimension in the $n$-point function; these make the diagrams more divergent. However, in this way $D = 4L - 2I$ not only counts the degree of divergence, but also the **powers in the external momenta and masses**. As long as the momenta and masses are small, diagrams with higher $D$ (i.e., higher loop diagrams) will be more and more suppressed. If we supply enough renormalization conditions at a given order $D$ to remove the infinities, we can stop the perturbative expansion after a few terms. Thus, non-renormalizable theories can be viewed as low-energy effective field theories.

The situation is illustrated in Fig. 4.13, which collects the lowest perturbative loop diagrams contributing to a four-point function with $\phi^4$ and $\phi^6$ vertices. In the horizontal direction we increase $V_4$, the number of vertices corresponding to the renormalizable $\phi^4$ interaction, which does not increase $D$. In the vertical direction we increase the number of vertices $V_6$ of the non-renormalizable $\phi^6$ interaction. Each subsequent row increases $D$ by two, so its diagrams are suppressed by a power of two at small momenta and masses compared to those in the previous row.

On the other hand, there is no suppression in the horizontal direction where $D$ does not change; here we would need to rely on the smallness of $\lambda_4$ to stop the series. This is where **derivative couplings** come in. The nonlinear sigma model in (4.4.40) contains $\varphi^p$ interactions ($p = 4 + 2r$) with derivatives, which become powers of momenta $l^\mu$ in momentum space. The generic structure of the $\varphi^p$ couplings is

$$p = 4: \ \frac{l^2}{\Lambda^2}\,, \qquad p = 6: \ \frac{l^2}{\Lambda^4}\,, \qquad p = 8: \ \frac{l^2}{\Lambda^6}\,, \qquad \text{etc.} \tag{4.4.45}$$

Each vertex in the nonlinear sigma model has two powers of derivatives, which we denote by $d_\chi = 2$. In principle we could construct theories with higher powers of derivatives, e.g. $d_\chi = 4$; in that case, the couplings would have the form

$$p = 4: \ \frac{l^4}{\Lambda^4}\,, \qquad p = 6: \ \frac{l^4}{\Lambda^6}\,, \qquad p = 8: \ \frac{l^4}{\Lambda^8}\,, \qquad \text{etc.} \tag{4.4.46}$$

Eq. (4.4.42) still holds for derivative couplings since the mass dimensions of the $\varphi^p$ couplings are still $4 - p$. However, they are now the differences stemming from the numerators with momentum powers $d_\chi$ and denominators with powers of $\Lambda^2$, where the former contribute to the degree of divergence $D$. If we split $4-p = d_\chi - (d_\chi + p - 4)$, then for a theory with fixed $d_\chi$ we have

$$\sum_p (4-p)\, V_p = \sum_p \left[d_\chi - (\dots)\right] V_p = d_\chi V - (\dots)\,, \tag{4.4.47}$$

where the rest does not contribute to $D$ but only to the mass dimension $[\Gamma_n]$. Therefore we arrive at

$$D = 4L - 2I + d_\chi V, \tag{4.4.48}$$

which counts the powers in the external momenta and masses.

If we now go back to Fig. 4.13 and interpret the $\phi^4$ and $\phi^6$ interactions as derivative couplings with $d_\chi = 2$ like in the nonlinear sigma model, we see that $D$ not only increases vertically but also horizontally. Then up to $D = 4$, for instance, we only need to keep a small number of diagrams. The same diagrams in a theory with $d_\chi = 4$ would carry even larger $D$. For a general theory containing couplings with any possible $d_\chi$, Eq. (4.4.48) generalizes to

$$D = 4L - 2I + \sum_{d_\chi} d_\chi V^{(d_\chi)}\,, \qquad \sum_{d_\chi} V^{(d_\chi)} = V, \tag{4.4.49}$$

where $V^{(d_\chi)}$ is the number of vertices in a diagram from a given order in $d_\chi$. In this way, $D$ tells us where to stop the perturbative expansion: diagrams with higher $D$ become less and less important at low momenta and small masses. The assumption of 'small masses' is justified since the propagators in the loops are pions and their masses are indeed small.

We finally note that the three quantities $L$, $I$ and $V$ are not independent but related by $L + V = I + 1$. Thus we could substitute $I$ and write Eq. (4.4.48) as

$$D = 2 + 2L + (d_\chi - 2)V, \tag{4.4.50}$$

which shows directly that $D$ grows with the number of loops and vertices.

**Fermion Lagrangian.** We have not yet addressed the fermionic part

$$\mathcal{L}_N = \overline{\psi}\left(i\partial\!\!\!/ - g\Lambda\,\Omega\right)\psi \tag{4.4.51}$$

of the Lagrangian (4.4.37), which contains the nucleon mass term through a complicated dependence on the pion fields encoded in $\Omega$. In analogy to Eq. (4.4.18), we rewrite the Dirac-flavor matrix $\Omega$ in terms of

$$\Sigma = \exp\left(i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\varphi}}{\Lambda}\right) = \cos z + i\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,\frac{\sin z}{\Lambda z}\,, \tag{4.4.52}$$

which is a matrix in $SU(2)$ flavor space only. Here we used the exponential representation $\alpha(z) = z$. If we use the chiral projectors $\mathsf{P}_\omega$ and the right- and left-handed spinors $\psi_\omega$, $\overline{\psi}_\omega$, with $\omega = \pm$, and follow the same steps as in (4.4.18) and below, we find

$$\Omega = \mathsf{P}_+\Sigma\,\mathsf{P}_+ + \mathsf{P}_-\Sigma^\dagger\,\mathsf{P}_- \quad\Rightarrow\quad \overline{\psi}\,\Omega\,\psi = \overline{\psi}_-\Sigma\,\psi_+ + \overline{\psi}_+\,\Sigma^\dagger\,\psi_- \tag{4.4.53}$$

as well as

$$|\partial_\mu\Omega|^2 = |\partial_\mu\Sigma|^2 = \tfrac{1}{2}\,\mathrm{Tr}\left\{\partial_\mu\Sigma^\dagger\,\partial^\mu\Sigma\right\}. \tag{4.4.54}$$

The chiral invariance of $\mathcal{L}_N$ implies the following transformation behavior under the group $SU(2)_L\times SU(2)_R$, where $U_-$ and $U_+$ are left- and right-handed transformation matrices with independent group parameters:

$$\Sigma' = U_-\Sigma\,U_+^\dagger\,. \tag{4.4.55}$$

Note that in the literature it is common to write

$$\psi_- = \psi_L\,,\quad \psi_+ = \psi_R\,,\quad U_- = L\,,\quad U_+ = R\,,\quad \Sigma = U\,. \tag{4.4.56}$$

Next, we redefine the fermion fields such that $\overline{\psi}\,\Omega\,\psi$ becomes a simple mass term. To do so, we introduce the $SU(2)$ matrices

$$\xi_\pm(x) = \exp\left(\pm i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\varphi}}{2\Lambda}\right) \quad\Rightarrow\quad \xi_\omega^\dagger = \xi_{-\omega}\,,\quad \Sigma = \xi_+\xi_+\,,\quad \Sigma^\dagger = \xi_-\xi_-\,, \tag{4.4.57}$$

and insertion in Eq. (4.4.53) yields

$$\Omega = \sum_\omega \mathsf{P}_\omega\,\xi_\omega\,\xi_\omega\,\mathsf{P}_\omega \quad\Rightarrow\quad \overline{\psi}\,\Omega\,\psi = \sum_\omega \overline{\psi}_{-\omega}\,\xi_\omega\,\xi_\omega\,\psi_\omega\,. \tag{4.4.58}$$

Defining the spinors

$$\begin{aligned}\Psi_\omega &= \xi_\omega\,\psi_\omega = \xi_\omega\,\mathsf{P}_\omega\,\psi\,,\\ \overline{\Psi}_\omega &= \overline{\psi}_\omega\,\xi_{-\omega} = \overline{\psi}\,\mathsf{P}_{-\omega}\,\xi_{-\omega}\,,\end{aligned}\qquad \Psi = \sum_\omega \Psi_\omega \quad\Rightarrow\quad \begin{aligned}\Psi_\omega &= \mathsf{P}_\omega\,\Psi\,,\\ \overline{\Psi}_\omega &= \overline{\Psi}\,\mathsf{P}_{-\omega}\,,\end{aligned} \tag{4.4.59}$$

we arrive at

$$\overline{\psi}\,\Omega\,\psi = \sum_\omega \overline{\Psi}_{-\omega}\,\Psi_\omega = \overline{\Psi}\sum_\omega \mathsf{P}_\omega^2\,\Psi = \overline{\Psi}\sum_\omega \mathsf{P}_\omega\,\Psi = \overline{\Psi}\,\Psi\,. \tag{4.4.60}$$

In turn, the kinetic term $\overline{\psi}\, i\partial\!\!\!/\, \psi$ becomes more complicated:

$$\begin{aligned}\overline{\psi}\, i\partial\!\!\!/\, \psi &= \sum_\omega \overline{\psi}_\omega\, i\partial\!\!\!/\, \psi_\omega = \sum_\omega \overline{\Psi}_\omega\, \xi_\omega\, i\partial\!\!\!/\, \xi_{-\omega}\, \Psi_\omega = \overline{\Psi} \Big( \sum_\omega \xi_\omega\, i\partial\!\!\!/\, \xi_{-\omega}\, \mathsf{P}_\omega \Big) \Psi \\ &= \overline{\Psi} \Big( i\partial\!\!\!/ + \sum_\omega i\xi_\omega \,(\partial\!\!\!/ \xi_{-\omega})\, \mathsf{P}_\omega \Big) \Psi = \overline{\Psi} \left( i\partial\!\!\!/ + v\!\!\!/ + a\!\!\!/ \gamma_5 \right) \Psi.\end{aligned} \tag{4.4.61}$$

In the last step we introduced the vector and axialvector fields

$$\begin{aligned} v^\mu &= \frac{i}{2}\,[\xi_+, \partial^\mu \xi_-] = \frac{i}{2}\,(\xi_+\, \partial^\mu \xi_- + \xi_-\, \partial^\mu \xi_+)\,, \\ a^\mu &= \frac{i}{2}\,\{\xi_+, \partial^\mu \xi_-\} = \frac{i}{2}\,(\xi_+\, \partial^\mu \xi_- - \xi_-\, \partial^\mu \xi_+)\,, \end{aligned} \tag{4.4.62}$$

where we used $\partial_\mu\,(\xi_- \xi_+) = 0$. Their combination gives

$$v\!\!\!/ + a\!\!\!/ \gamma_5 = i \left( \xi_+\, \partial\!\!\!/ \xi_-\, \mathsf{P}_+ + \xi_-\, \partial\!\!\!/ \xi_+\, \mathsf{P}_- \right), \tag{4.4.63}$$

which is the combination that appears in Eq. (4.4.61).

Putting everything together, the Lagrangian (4.4.37) takes the form

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_\pi = \overline{\Psi} \left( i\partial\!\!\!/ + v\!\!\!/ + a\!\!\!/ \gamma_5 - M \right) \Psi + \frac{\Lambda^2}{4}\, \mathrm{Tr} \left\{ \partial_\mu \Sigma^\dagger\, \partial^\mu \Sigma \right\}, \tag{4.4.64}$$

where the nucleon mass is $M = g\Lambda$ and the original Yukawa couplings between the nucleon and the pion have turned into vector and axialvector couplings. Expanding $v^\mu$ and $a^\mu$ in the lowest powers of the pion fields, we obtain

$$v_\mu = -\frac{1}{4\Lambda^2}\, \boldsymbol{\tau} \cdot (\boldsymbol{\varphi} \times \partial_\mu\, \boldsymbol{\varphi}) + \dots\,, \qquad a_\mu = \frac{1}{2\Lambda}\, \boldsymbol{\tau} \cdot \partial_\mu\, \boldsymbol{\varphi} + \dots\,. \tag{4.4.65}$$

The **axialvector coupling** of the pion to the nucleon induced by $a^\mu$ corresponds to the second diagram in Fig. 4.12. The third diagram is the **Weinberg-Tomozawa term** stemming from $v^\mu$, a seagull-like contact interaction between two pions and the nucleon which gives the dominant tree-level contribution to $N\pi$ scattering.

Moreover, with $\Sigma = \xi_+ \xi_+$ it follows that the transformation behavior $\Sigma' = U_- \Sigma\, U_+^\dagger$ is satisfied if $\xi_+$ transforms like

$$\xi_+' = U_- \xi_+ K^\dagger \equiv K\, \xi_+\, U_+^\dagger \quad \Rightarrow \quad \xi_-' = U_+ \xi_- K^\dagger = K\, \xi_-\, U_-^\dagger\,, \tag{4.4.66}$$

where $K$ is a unitary $SU(2)$ matrix which depends on $U_+$, $U_-$ but also on the pion fields $\varphi_a$ themselves which carry a dependence on $x$. As a result, we find

$$\Psi_\omega' = K\, \Psi_\omega\,, \qquad \Psi' = K\, \Psi\,, \qquad \begin{aligned} v_\mu' &= K v_\mu K^\dagger + iK(\partial_\mu K^\dagger)\,, \\ a_\mu' &= K a_\mu K^\dagger\,. \end{aligned} \tag{4.4.67}$$

If we define the **chiral covariant derivative** by $D_\mu = \partial_\mu - iv_\mu$, the comparison with Eq. (2.1.5) shows that the transformation of $v_\mu$ is that of a vector field under a *local* symmetry. In other words, instead of a global invariance with respect to $U_+$ and $U_-$, chiral symmetry has turned into a local invariance under a transformation with $K$.

**Explicit symmetry breaking.** In order to make contact with QCD, we add the following term to the Lagrangian of the nonlinear sigma model:

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_\pi + \mathcal{L}_{\text{sb}}\,, \qquad \mathcal{L}_{\text{sb}} = \frac{b\,\Lambda^3}{2}\,\text{Tr}\left(\mathsf{M}\,\Sigma^\dagger + \Sigma\,\mathsf{M}^\dagger\right). \tag{4.4.68}$$

It depends on the two-flavor quark mass matrix $\mathsf{M}$ from Eq. (3.1.15), and $b$ is a dimensionless parameter. We can compare this to $\mathcal{L}_{QCD}|_{\text{massless}} - \bar{\psi}\,\mathsf{M}\,\psi$ based on the following arguments:

- $\mathcal{L}_{\text{sb}}$ has mass dimension four and is linear in the quark mass matrix.
- $\mathcal{L}_{\text{sb}}$ breaks chiral symmetry explicitly. To see this, consider equal quark masses $\mathsf{M} = m\,\mathbb{1}$; then from $\Sigma' = U_-\Sigma\,U_+^\dagger$ we have

$$\text{Tr}\left(\Sigma'^\dagger + \Sigma'\right) = \text{Tr}\left(U_+\Sigma\,U_-^\dagger + U_-\Sigma\,U_+^\dagger\right). \tag{4.4.69}$$

  Recall from Eq. (3.1.44) that a $SU(2)_V$ transformation implies $\varepsilon_+ = \varepsilon_-$ and thus $U_+ = U_-$, whereas a $SU(2)_A$ transformation implies $\varepsilon_+ = -\varepsilon_-$ and $U_+ = U_-^\dagger$. Therefore, $\mathcal{L}_{\text{sb}}$ is still invariant under isospin symmetry $SU(2)_V$ but it breaks $SU(2)_A$, as does the mass term in the QCD Lagrangian.
- Eq. (4.4.52) tells us that

$$\Sigma + \Sigma^\dagger = 2\cos z = 2\cos\frac{\boldsymbol{\varphi}^2}{\Lambda^2} = 2\left[1 - \frac{\boldsymbol{\varphi}^2}{2\Lambda^2} + \dots\right], \tag{4.4.70}$$

  and because $\mathsf{M} = \mathsf{M}^\dagger$ we find

$$\mathcal{L}_{\text{sb}} = b\,\Lambda^3\,(m_u + m_d)\left[1 - \frac{\boldsymbol{\varphi}^2}{2\Lambda^2} + \dots\right] = -\frac{1}{2}\,b\,\Lambda\,(m_u + m_d)\,\boldsymbol{\varphi}^2 + \dots \tag{4.4.71}$$

  Hence we can identify the **pion mass** from $m_\pi^2 = b\,\Lambda\,(m_u + m_d)$.
- From the mass term in the QCD Lagrangian we can infer the VEV of the Hamiltonian density

$$\langle\mathcal{H}_{QCD}\rangle = \langle\bar{\psi}\,\mathsf{M}\,\psi\rangle = m_u\,\langle\bar{u}u\rangle + m_d\,\langle\bar{d}d\rangle\,, \tag{4.4.72}$$

  from where we obtain the quark condensate by taking the derivative of $\langle\mathcal{H}_{QCD}\rangle$ with respect to any of the quark masses and setting $m_q = 0$. Comparison with the effective Hamiltonian at vanishing meson fields ($\Sigma = 1$) yields

$$\langle\mathcal{H}_{\text{sb}}\rangle = -b\,\Lambda^3\,(m_u + m_d) \quad \Rightarrow \quad -b\,\Lambda^3 = \langle\bar{u}u\rangle = \langle\bar{d}d\rangle\,, \tag{4.4.73}$$

  which suggests the identification of $b$ with the dimensionless **quark condensate**. Comparing this with $m_\pi^2$ from above, we arrive at

$$\Lambda^2\,m_\pi^2 = -\frac{m_u + m_d}{2}\,\langle\bar{u}u + \bar{d}d\rangle\,, \tag{4.4.74}$$

  which is identical to the GMOR relation (4.2.41) if we identify the scale $\Lambda$ with the chiral-limit **pion decay constant** $f_\pi$.

We can establish more analogies if we derive the **vector and axialvector currents** and their divergences. Usually we would do this via Eq. (3.1.2) by taking the derivative of the Lagrangian with respect to the derivative of the fields. However, this can become cumbersome if $\mathcal{L}$ depends on the fields in a complicated way. A simpler method is to consider the variation of the (globally invariant) action under a *local* gauge transformation, and evaluate it for the solutions of the classical equations of motion:

$$\delta S = -\int d^4x\, \partial_\mu \sum_a \varepsilon_a\, j_a^\mu = -\int d^4x \sum_a (\partial_\mu \varepsilon_a\, j_a^\mu + \varepsilon_a\, \partial_\mu j_a^\mu)\,. \tag{4.4.75}$$

The variation is then no longer zero because the Lagrangian is not locally invariant, i.e., the surface term is nonvanishing. On the other hand, in this way we can read off both the currents and their divergences (which vanish if the global symmetry is intact) as the coefficients of $\partial_\mu \varepsilon_a$ and $\varepsilon_a$.

To compute the variation of the Lagrangian $\mathcal{L}_\pi = \frac{\Lambda^2}{4} \operatorname{Tr}\left\{\partial_\mu \Sigma^\dagger\, \partial^\mu \Sigma\right\}$, we work out the infinitesimal transformation of the meson matrix $\Sigma$,

$$\begin{aligned} \Sigma' &= U_- \Sigma\, U_+^\dagger = (1 + i\varepsilon_-)\, \Sigma\, (1 - i\varepsilon_+) = 1 + i\varepsilon_-\, \Sigma - i \Sigma\, \varepsilon_+ \\ &= 1 + i[\varepsilon_V, \Sigma] - i\{\varepsilon_A, \Sigma\} = 1 + \delta\Sigma\,, \end{aligned} \tag{4.4.76}$$

where we used the infinitesimal relation $\varepsilon_\pm = \varepsilon_V \pm \varepsilon_A$ from Eq. (3.1.44). After some algebra, the currents can be read off from the coefficients of $\partial_\mu \varepsilon_V^a$ and $\partial_\mu \varepsilon_A^a$ in $\delta S$:

$$\begin{aligned} V_a^\mu &= -i\,\frac{\Lambda^2}{4} \operatorname{Tr}\left(\tau_a \left[\Sigma, \partial^\mu \Sigma^\dagger\right]\right) = \varepsilon_{abc}\, \varphi_b\, \partial^\mu \varphi_c + \dots\,, \\ A_a^\mu &= i\,\frac{\Lambda^2}{4} \operatorname{Tr}\left(\tau_a \left\{\Sigma, \partial^\mu \Sigma^\dagger\right\}\right) = \Lambda\, \partial^\mu \varphi_a + \dots \end{aligned} \tag{4.4.77}$$

Vice versa, the coefficients of $\varepsilon_V^a$ and $\varepsilon_A^a$ vanish and therefore the currents are conserved: $\partial_\mu V_a^\mu = \partial_\mu A_a^\mu = 0$.

On the other hand, the variation of the symmetry-breaking mass term $\mathcal{L}_{\text{sb}}$ is

$$\begin{aligned} &\delta\Sigma + \delta\Sigma^\dagger = i\big[\varepsilon_V, \Sigma + \Sigma^\dagger\big] - i\big\{\varepsilon_A, \Sigma - \Sigma^\dagger\big\} = \frac{2}{\Lambda} \sum_a \varepsilon_A^a\, \varphi_a + \dots \\ &\Rightarrow\ \delta\mathcal{L}_{\text{sb}} = \sum_a \varepsilon_A^a\, b\, \Lambda^2\, (m_u + m_d)\, \varphi_a\,, \end{aligned} \tag{4.4.78}$$

in which case the divergences of the currents become

$$\partial_\mu V_a^\mu = 0\,, \qquad \partial_\mu A_a^\mu = -b\, \Lambda^2\, (m_u + m_d)\, \varphi_a = -\Lambda\, m_\pi^2\, \varphi_a\,. \tag{4.4.79}$$

This is the analogue of the PCAC relation (3.1.39). If we take the divergence of the current in Eq. (4.4.77), we get back the classical equation of motion for the pion field, the Klein-Gordon equation:

$$\partial_\mu A_a^\mu = \Lambda\, \Box\, \varphi_a = -\Lambda\, m_\pi^2\, \varphi_a \quad \Rightarrow \quad (\Box + m_\pi^2)\, \varphi_a = 0\,. \tag{4.4.80}$$

## 4.4.2 Chiral perturbation theory

The approach developed so far looks promising, but it is also not quite satisfactory: We have eliminated the scalar meson in the linear sigma model at the price of a non-renormalizable low-energy effective theory. How is this better than the original approach? After all, it is still just a model that contains certain chosen interactions.

The idea of **chiral perturbation theory (ChPT)**, formulated by Weinberg and then applied by Gasser and Leutwyler, is the following: we do not know the underlying microscopic interactions that constitute hadronic $n-$point functions, so instead of providing a specific model for them (like the linear or nonlinear sigma model) we write down a systematic expansion in *all* possible terms that are compatible with the symmetries of QCD. The resulting theory is an effective theory formulated in terms of nucleons and pions, and eventually also other $SU(3)$ multiplet members, but since it contains all possible interactions it is a low-energy expansion of QCD.

This theory is non-renormalizable and the resulting Lagrangian will contain infinitely many terms with infinitely many free parameters. However, this is not a serious issue as long as we stick to the lowest orders in the expansion in derivatives and pion masses (i.e., we work at small momenta and close to the chiral limit). If we can fix a small number of unknown parameters — the **low-energy constants (LECs)** — from experiment or lattice QCD, we should be able to make a range of predictions already at tree level or at a low loop orders, e.g. for $\pi\pi$ or $N\pi$ scattering amplitudes, electromagnetic and weak interaction processes, etc.

With the meson matrix $\Sigma = e^{i\boldsymbol{\tau}\cdot\boldsymbol{\varphi}/\Lambda}$, the quark mass matrix $\mathsf{M}$ and the scale $\Lambda$ (identified with the chiral-limit pion decay constant $f_\pi$) as building blocks, we can organize the infinitely many possible terms in the Lagrangian by their number of derivatives and powers of pion masses. Because of Lorentz invariance, each term in the Lagrangian must contain an even number $d_\chi = 2, 4, 6, \dots$ of derivatives, and there can be no derivative-free term because $\text{Tr}\,\{\Sigma^\dagger\Sigma\}$ is a constant. We can then write

$$\mathcal{L} = \sum_{d_\chi} \mathcal{L}^{(d_\chi)} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots \tag{4.4.81}$$

Because each derivative translates to a factor of momentum when taking matrix elements, low orders in derivatives correspond to small momenta and larger-derivative terms only have a small effect. On the other hand, each instance of the mass matrix $\mathsf{M}$ becomes a factor $\propto m_\pi^2$, as we saw for the lowest-order term in Eq. (4.4.71), and therefore also enters at order $d_\chi = 2$. The lowest-order terms are then given by

$$\begin{aligned}
&\bullet\ \mathcal{L}^{(2)} : \text{Tr}\,\{\partial_\mu\Sigma^\dagger\,\partial^\mu\Sigma\},\ \text{Tr}\,\{\mathsf{M}\,\Sigma^\dagger + \Sigma\,\mathsf{M}^\dagger\}\\
&\bullet\ \mathcal{L}^{(4)} : \big(\text{Tr}\,\{\partial_\mu\Sigma^\dagger\,\partial^\mu\Sigma\}\big)^2,\ \text{Tr}\,\{\partial_\mu\Sigma^\dagger\,\partial_\nu\Sigma\}\,\text{Tr}\,\{\partial^\mu\Sigma^\dagger\,\partial^\nu\Sigma\}\,,\\
&\qquad\quad \text{Tr}\,\{\partial_\mu\Sigma^\dagger\,\partial_\nu\Sigma\}\,\text{Tr}\,\{\mathsf{M}\,\Sigma^\dagger + \Sigma\,\mathsf{M}^\dagger\}\,,\ \big(\text{Tr}\,\{\mathsf{M}\,\Sigma^\dagger \pm \Sigma\,\mathsf{M}^\dagger\}\big)^2,\\
&\qquad\quad \text{Tr}\,\{\mathsf{M}\,\Sigma^\dagger\,\mathsf{M}\,\Sigma^\dagger + \mathsf{M}^\dagger\,\Sigma\,\mathsf{M}^\dagger\,\Sigma\}\,,\ \text{Tr}\,\{\mathsf{M}^\dagger\mathsf{M}\}\\
&\bullet\ \mathcal{L}^{(6)} :\ \dots
\end{aligned} \tag{4.4.82}$$

The lowest-order Lagrangian for $d_\chi = 2$ is just the nonlinear sigma model.

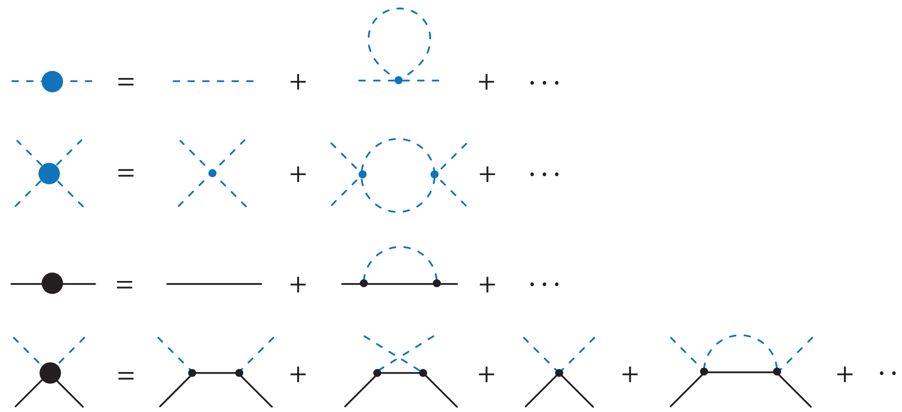

FIG. 4.14: Chiral expansion of $\pi$ and $N$ propagators, $\pi\pi$ and $N\pi$ scattering.

Unfortunately, the inclusion of nucleons disrupts the power counting because the nucleon mass is not a small scale, and loop diagrams may contribute at the same order as tree-level diagrams. This problem is addressed in **heavy-baryon ChPT**, where manifest covariance is traded for a systematic power counting.

One can then perform loop expansions[5] for amplitudes like those in Fig. 4.14:

- $\pi$ and $N$ propagators, where the latter allow one to determine the nucleon mass as a function of $m_\pi^2$.
- $\pi\pi$ scattering near pion-production threshold ($s = 4m_\pi^2$, $t = u = 0$), which is the onset of the physical region. At threshold, the scattering amplitude is expressed by the $\pi\pi$ **scattering lengths**, which vanish in the chiral limit.
- $N\pi$ scattering close to pion production threshold $s = (M + m_\pi)^2$, where at the threshold one can extract the $N\pi$ scattering lengths; etc.

Several ChPT extensions are possible:

- In the case of $SU(3)_f$, the meson matrix becomes $\Sigma = e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\varphi}/\Lambda}$, where the $\lambda_a$ are the Gell-Mann matrices:

$$\boldsymbol{\lambda}\cdot\boldsymbol{\varphi} = \sqrt{2}\begin{pmatrix} \frac{\pi_0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi_0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -2\frac{\eta_8}{\sqrt{6}} \end{pmatrix}. \qquad (4.4.83)$$

- The effect of the axial anomaly can be implemented through a **Wess-Zumino-Witten (WZW)** term.
- Since ChPT is still a *perturbation theory* around small momenta and pion masses, it cannot generate resonance poles which are nonperturbative effects. This is addressed in **unitarized ChPT**, which amounts to solving self-consistent Bethe-Salpeter equations of the form (3.1.153) but for hadronic correlation functions.

[5]See e.g. S. Scherer, *Adv. Nucl. Phys.* **27** (2003) 277, hep-ph/0210398 for explicit calculations.