4.3 $U(1)_A$ anomaly

We have seen that spontaneous chiral symmetry breaking should affect all axial symmetries including the flavor-singlet $U(1)_A$. The fact that there is no good candidate for a flavor-singlet (pseudo-) Goldstone boson in the spectrum is related to the anomalous $U(1)_A$ breaking. **Anomalies** are symmetries of classical Lagrangians that are broken at the quantum level. They arise when regularization destroys a symmetry and there is no regulator choice that can preserve it. Since the symmetry is lost, there is no Goldstone boson because the quantum corrections generate a mass for that mode.

Anomalies are again a typical feature of **axial symmetries**. In contrast to spontaneous symmetry breaking, where the symmetry is lost due to dynamical effects, anomalies have their origin in short-distance singularities of the currents $A^\mu_a = \overline{\psi}\,\gamma^\mu\gamma_5\,\mathsf{t}_a\,\psi$ and $A^\mu = \overline{\psi}\,\gamma^\mu\gamma_5\,\psi$. These are composite operators at the same space-time point which are potentially divergent and have to be regularized. In principle, the problem would also affect vector currents, but in that case it is possible to find appropriate regularization prescriptions that leave their symmetry intact. Vector symmetries are related to conserved charges (color charge, electromagnetic charge, flavor charges, etc.). If they were broken at the quantum level, we would not only lose charge conservation but also gauge symmetry, and the theory would become nonrenormalizable and inconsistent. In this sense, global axial symmetries are 'less important' and the fact that they produce anomalies is not a serious problem for the theory. (Except when they are also promoted to gauge symmetries: if a gauge symmetry is broken anomalously, then one needs anomaly cancellations between different sectors of the theory.)

In the following we will see that

- QCD only leads to an anomalous $U(1)_A$ breaking, which has observable consequences for the η and η' masses, whereas
- QED also induces an anomalous $SU(N_f)_A$ breaking, which can be observed in the $\pi^0 \to \gamma\gamma$ decay.

We already wrote down the basic relations that characterize the anomalous $U(1)_A$ breaking in QCD. We have anticipated in Eq. (3.1.54) that the divergence of the axialvector singlet current picks up an anomalous contribution

$$\partial_\mu A^\mu = 2i\,\overline{\psi}\,\mathsf{M}\,\gamma_5\,\psi + N_f\,\mathcal{Q}(x)\,, \tag{4.3.1}$$

where $\mathcal{Q}(x)$ is the topological charge density that we encountered in Section 2.1:

$$\mathcal{Q}(x) = \frac{g^2}{8\pi^2}\,\mathrm{Tr}\,\{\widetilde{F}_{\mu\nu}\,F^{\mu\nu}\,\}, \qquad \widetilde{F}^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,. \tag{4.3.2}$$

The derived relation (3.2.42) entails that the mass of the η_0 does not vanish in the chiral limit, so there is no flavor-singlet Goldstone boson:

$$f_{\eta_0} m_{\eta_0}^2 = 2\,\frac{m_u+m_d+m_s}{3}\,r_{\eta_0} + \frac{g^2 N_f}{(4\pi)^2}\,\langle 0|\,\widetilde{F}^{\mu\nu}_a(0)\,F^a_{\mu\nu}(0)\,|\eta_0\rangle\,. \tag{4.3.3}$$

Anomalies from the path integral. To see how the anomalous term comes about, suppose we start from an action $S[\psi,\overline{\psi}]$ that is invariant under global $U(1)_A$ transformations, e.g. the fermionic part of the Lagrangian for massless quarks:

$$\mathcal{L} = \overline{\psi}\, i\partial\!\!\!/\, \psi\,, \qquad \psi' = e^{i\varepsilon\gamma_5}\psi\,, \qquad \overline{\psi}' = \overline{\psi}\, e^{i\varepsilon\gamma_5}\,. \tag{4.3.4}$$

To derive WTIs for global flavor symmetries from the path integral, we need to employ the **background-field method** discussed below Eq. (3.1.108): we add a source term to the action with a background field B^μ, so that the total action that enters in the partition function is *locally* invariant by construction:

$$Z[B] = \int \mathcal{D}[\psi,\overline{\psi}]\, e^{i\left(S[\psi,\overline{\psi}]+\widetilde{S}[\psi,\overline{\psi},B]\right)}\,. \tag{4.3.5}$$

This means we need to impose a $U(1)_A$ transformation behavior for the B^μ field with a covariant derivative:

$$B'_\mu = B_\mu + \frac{1}{g}\,\partial_\mu\varepsilon\,, \qquad D_\mu = \partial_\mu - igB_\mu\gamma_5\,. \tag{4.3.6}$$

The resulting Lagrangian

$$\overline{\psi}\, iD\!\!\!\!/\, \psi = \overline{\psi}\,(i\partial\!\!\!/ + gB\!\!\!\!/\gamma_5)\,\psi = \mathcal{L} + gA_\mu B^\mu \tag{4.3.7}$$

is locally invariant as desired. As before, A_μ is the $U(1)_A$ axialvector current and not the gluon field (in the following we denote the gluon fields by $\mathbf{A}^\mu$ to avoid confusion) and the extra source term in the action is

$$\widetilde{S}[\psi,\overline{\psi},B] = g\int d^4x\, A_\mu B^\mu\,, \qquad A^\mu = \overline{\psi}\,\gamma^\mu\gamma_5\,\psi\,. \tag{4.3.8}$$

Because all terms in the path integral are locally gauge invariant, a gauge transformation $\{\psi,\overline{\psi},B\} \to \{\psi',\overline{\psi}',B'\}$ does not change the partition function: $Z[B]=Z[B']$. If we then relabel the quark fields back to unprimed ones and work out the transformation of B only, we find

$$Z[B'] = \int \mathcal{D}[\psi,\overline{\psi}]\, e^{i\left(S[\psi,\overline{\psi}]+\widetilde{S}[\psi,\overline{\psi},B]+\delta\widetilde{S}\right)} = Z[B]\,\langle e^{i\delta\widetilde{S}}\rangle_B \tag{4.3.9}$$

and therefore $\langle\delta\widetilde{S}\rangle_B = 0$. Then, with

$$\delta\widetilde{S} = \int d^4x\, A_\mu(x)\,\partial^\mu\varepsilon(x) = -\int d^4x\,\varepsilon(x)\,\partial_\mu A^\mu(x) \tag{4.3.10}$$

we arrive at the usual PCAC relation for the flavor-singlet case:

$$\langle\partial_\mu A^\mu\rangle_B = 0\,. \tag{4.3.11}$$

This means that current conservation holds inside the vacuum expectation value in the presence of the background field. Note that without it the relation would be trivial: $\langle\partial_\mu A^\mu\rangle = \partial_\mu\,\langle A^\mu\rangle = 0$ because $\langle A^\mu\rangle = 0$. If we had also included source terms η, $\bar{\eta}$ for the quarks, we would have obtained the usual WTIs for the n-point functions like in Eq. (3.1.111).

But where is the *anomalous* term? As always we assumed that the **path integral measure** remains invariant under the transformation. However, for axial transformations this is not necessarily the case. The origin of this behavior is the transformation of the Dirac spinors

$$\psi'(x) = e^{+i\varepsilon\gamma_5}\psi(x)\,, \qquad \bar{\psi}'(x) = \bar{\psi}(x)\,e^{+i\varepsilon\gamma_5}\,, \tag{4.3.12}$$

which leads to a Jacobian determinant of the transformation:

$$\mathcal{D}[\psi',\bar{\psi}'] = (\det C)^{-2}\,\mathcal{D}[\psi,\bar{\psi}]\,. \tag{4.3.13}$$

It turns out that this determinant is ill-defined $(0\cdot\infty)$ and requires regularization, which in turn breaks the $U(1)_A$ symmetry. The final result is just the anomalous term:

$$(\det C)^{-2} = \exp\left(-i\int d^4x\,\varepsilon(x)\,N_f\,\mathcal{Q}(x)\right). \tag{4.3.14}$$

As a consequence, $Z[B'] \neq Z[B]$ under a gauge transformation but instead

$$Z[B'] = Z[B]\left\langle \exp\left(-i\int d^4x\,\varepsilon(x)\,N_f\,\mathcal{Q}(x)\right)\right\rangle_B\,, \tag{4.3.15}$$

and comparison with Eq. (4.3.9) gives the anomalous correction to the PCAC relation:

$$\langle \partial_\mu A^\mu - N_f\,\mathcal{Q}\rangle_B = 0\,. \tag{4.3.16}$$

Fujikawa's method. In order to prove Eq. (4.3.14), let us expand the functional determinant into eigenfunctions of the **Dirac operator** $\not{D} = \not{\partial} - ig\not{\mathbf{A}}$. This is now again the usual covariant derivative with the gluon field and not the quantity in Eq. (4.3.6), which we no longer need. Assume that the Dirac operator $\not{D}$ is hermitian, so that it has real eigenvalues λ_n and a set of orthonormal, complete eigenfunctions:

$$\not{D}\,\varphi_n(x) = \lambda_n\,\varphi_n(x)\,, \qquad \begin{aligned} \int d^4x\,\varphi^\dagger_{m,i}(x)\,\varphi_{n,j}(x) &= \delta_{mn}\,\delta_{ij}\,, \\ \textstyle\sum_n \varphi_{n,i}(x)\,\varphi^\dagger_{n,j}(y) &= \delta^4(x-y)\,\delta_{ij}\,, \end{aligned} \tag{4.3.17}$$

where i,j collect the Dirac, color and flavor indices. To ensure the (anti-) hermiticity of the Dirac operator, we should really do this in Euclidean space, but let us ignore this subtlety in what follows.

We can expand the spinors ψ, $\bar{\psi}$ into these eigenfunctions, where the coefficients a_n and $\bar{b}_n$ are independent Grassmann variables, and write down the path integral measure:

$$\psi(x) = \sum_n a_n\,\varphi_n(x)\,, \quad \bar{\psi}(x) = \sum_n \varphi^\dagger_n(x)\,\bar{b}_n\,, \quad \mathcal{D}[\psi,\bar{\psi}] = \prod_n da_n \prod_m d\bar{b}_m\,. \tag{4.3.18}$$

As a side remark, the fermionic path integral can be written as the determinant of the Dirac operator (which is useful in lattice calculations):

$$\det\not{D} = \int \mathcal{D}[\psi,\bar{\psi}]\,e^{i\int d^4x\,\bar{\psi}\,i\not{D}\,\psi} = \int \prod_n da_n\,d\bar{b}_n\,e^{-\sum_n \bar{b}_n\,\lambda_n\,a_n} = \prod_n \lambda_n\,. \tag{4.3.19}$$

Now, if we use the orthogonality relation to project out the coefficients, an axial transformation changes a_n and $\bar{b}_n$ to

$$a'_n = \int d^4x\, \varphi_n^\dagger(x)\, \psi'(x) = \sum_m \underbrace{\int d^4x\, \varphi_n^\dagger(x)\, e^{i\varepsilon(x)\gamma_5}\, \varphi_m(x)}_{=:C_{nm}}\, a_m \tag{4.3.20}$$

so that we have

$$a'_n = \sum_m C_{nm}\, a_m\,, \qquad \bar{b}'_m = \sum_n C_{nm}\, \bar{b}_n\,. \tag{4.3.21}$$

Note that because we are dealing with axial transformations, both a_n and $\bar{b}_m$ transform with the same C_{mn},

$$C_{mn} = \delta_{mn} + i\int d^4x\, \varepsilon(x)\, \varphi_n^\dagger(x)\, \gamma_5\, \varphi_m(x) + \ldots, \tag{4.3.22}$$

and because the Grassmann measure transforms with the inverse determinant we arrive at Eq. (4.3.13). Using $\det C = e^{\mathrm{Tr}\,\ln C}$ and expanding the logarithm, we obtain

$$(\det C)^{-2} = \exp\left(-2i\int d^4x\, \varepsilon(x) \sum_n \varphi_n^\dagger(x)\, \gamma_5\, \varphi_n(x)\right), \tag{4.3.23}$$

which involves the 'functional trace' over γ_5. With the completeness relation in (4.3.17), the sum becomes

$$\sum_n \varphi_n^\dagger(x)\, \gamma_5\, \varphi_n(x) = \lim_{y\to x} \sum_n \varphi_{n,i}^\dagger(y)\, (\gamma_5)_{ij}\, \varphi_{n,j}(x) = \lim_{y\to x}\, \mathrm{Tr}\,\{\gamma_5\}\, \delta^4(x-y)\,, \tag{4.3.24}$$

where the trace goes over Dirac, color and flavor indices. The color-flavor trace gives a factor $N_f N_c$, whereas the Dirac trace vanishes but the δ-function diverges. Thus we have a $0\cdot\infty$ situation: this expression is possibly finite, but it is not well-defined and must be regulated.

Fujikawa suggested to regulate it in a gauge-invariant way by damping the contribution from the large eigenvalues by a Gaussian cutoff, with a regulator mass M that is taken to infinity in the end:

$$\begin{aligned} &\lim_{M\to\infty} \sum_n \varphi_n^\dagger(x)\, \gamma_5\, e^{-(\lambda_n/M)^2}\, \varphi_n(x) \\ &\quad = \lim_{M\to\infty} \sum_n \varphi_n^\dagger(x)\, \gamma_5\, e^{-(\not{D}/M)^2}\, \varphi_n(x) \\ &\quad = \lim_{\substack{M\to\infty \\ y\to x}} \mathrm{Tr}\left\{\gamma_5\, e^{-(\not{D}/M)^2}\right\} \delta^4(x-y) \\ &\quad = \lim_{M\to\infty} \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \mathrm{Tr}\left\{\gamma_5\, e^{-(\not{D}/M)^2}\right\} e^{ikx}\,. \end{aligned} \tag{4.3.25}$$

This regularization is gauge-invariant because the covariant derivative appears in it; hence, it preserves the vector gauge symmetry. To proceed, we express $\not{D}^2$ by

$$\begin{aligned} \not{D}^2 = \gamma^\mu\gamma^\nu D_\mu D_\nu &= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\, D_\mu D_\nu + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\, D_\mu D_\nu \\ &= D^2 + \frac{1}{4}[\gamma^\mu,\gamma^\nu]\,[D_\mu, D_\nu] = D^2 - \frac{ig}{4}[\gamma^\mu,\gamma^\nu]\, F_{\mu\nu} \end{aligned} \tag{4.3.26}$$

and exploit the relation (2.2.47),

$$e^{-ikx}\, f\left(\frac{\partial}{\partial x}\right) e^{ikx} = f\left(\frac{\partial}{\partial x} + ik\right), \tag{4.3.27}$$

where unsaturated derivatives vanish in the end. Eq. (4.3.25) then becomes

$$\cdots = \lim_{M\to\infty} \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left\{\gamma_5\, \exp\left(-\frac{(D+ik)^2}{M^2} + \frac{ig}{4M^2}\,[\gamma^\mu,\gamma^\nu]\, F_{\mu\nu}\right)\right\}. \tag{4.3.28}$$

When expanding the exponential, only terms with at least four γ matrices can survive the trace with γ_5, and only those $\propto 1/M^4$ which produce a dimensionless quantity after integration will survive the limit $M \to \infty$. These terms can only appear at quadratic order and produce

$$\frac{i}{4}\, \mathrm{Tr}\left\{\gamma_5\, \gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right\} = \varepsilon^{\mu\nu\alpha\beta}\,. \tag{4.3.29}$$

The resulting expression has the form

$$\cdots = \lim_{M\to\infty} \int \frac{d^4k}{(2\pi)^4} \left[e^{-\frac{k^2}{M^2}}\, \frac{g^2}{M^4}\, N_f\, \mathrm{Tr}\,\{\widetilde{F}_{\mu\nu}\, F^{\mu\nu}\,\} + \dots \right]. \tag{4.3.30}$$

Then, after integrating out the momentum k and sending $M \to \infty$, the final result becomes

$$\sum_n \varphi_n^\dagger(x)\, \gamma_5\, \varphi_n(x) = \lim_{y\to x}\, \mathrm{Tr}\,\{\gamma_5\}\, \delta^4(x-y) = \frac{g^2 N_f}{16\pi^2}\, \mathrm{Tr}\,\{\widetilde{F}_{\mu\nu}\, F^{\mu\nu}\,\}, \tag{4.3.31}$$

where only the color trace over the $SU(3)_C$ generators remains. Inserted in the determinant (4.3.23), we arrive at Eq. (4.3.14).

A few **remarks** are in order:

■ Note that we did not perform an 'additional renormalization' because the theory was already renormalized before. Renormalization means that the regulator remains in the theory, but it is hidden in the renormalization constants which must cancel each other in observables, together with the regulator dependence. Here we have merely cured a $0\cdot\infty$ situation by introducing a cutoff M that we sent to infinity at the end. However, the resulting finite expression has the property that it breaks the $U(1)_A$ symmetry. While we used exponential damping, one can show that this result is indeed independent of the chosen regularization as long as it is gauge invariant.

■ Since the topological charge is essentially the trace over γ_5, one can ask why only $U(1)_A$ and not the non-Abelian global $SU(N_f)_A$ transformations lead to anomalies. Repeating the analysis with $\varepsilon \to \sum_a \varepsilon_a\, \mathsf{t}_a$ yields

$$\partial_\mu A^\mu_a = \frac{g^2}{(4\pi)^2}\, \varepsilon^{\alpha\beta\mu\nu}\, F^b_{\alpha\beta}\, F^c_{\mu\nu}\, \mathrm{Tr}_F\,\{\mathsf{t}_a\}\, \mathrm{Tr}_C\,\{\mathsf{t}_b\, \mathsf{t}_c\}\,, \tag{4.3.32}$$

which vanishes in the flavor-octet case because $\mathrm{Tr}\{\mathsf{t}_a\} = 0$. In other words, gluons couple only to flavor-singlet currents, and the anomaly signals the breakdown of the $U(1)_A$ symmetry in the presence of gluons.

■ The topological charge density can be written as the divergence of a current, the **Chern-Simons current**:

$$\mathcal{Q}(x) = \partial_\mu K^\mu\,, \qquad K^\mu = \frac{g^2}{8\pi^2}\,\varepsilon^{\mu\nu\alpha\beta}\,\text{Tr}\left\{F_{\alpha\beta}\,\mathbf{A}_\nu + \frac{2ig}{3}\mathbf{A}_\alpha\mathbf{A}_\beta\mathbf{A}_\nu\right\}. \tag{4.3.33}$$

One could then conclude that the flavor-singlet PCAC relation (in the chiral limit) still induces a conserved current $\partial_\mu(A^\mu - N_f K^\mu) = 0$, which leads back to the argument that there should be a flavor-singlet Goldstone boson. However, K^μ and its corresponding charge $\int d^3x\, K^0$ are not gauge invariant, so they cannot couple to physical states and hence there is no conserved axial charge.

Triangle diagrams. The axial anomaly will show up (and was originally derived) in the calculation of correlation functions involving axialvector currents, e.g.

$$\langle 0|\mathsf{T}\, A^\mu(x)\, V^\alpha(y)\, V^\beta(z)\,|0\rangle\,, \quad \langle 0|\mathsf{T}\, A^\mu(x)\, A^\alpha(y)\, A^\beta(z)\,|0\rangle\,, \quad \text{etc.} \tag{4.3.34}$$

Take for example the WTI for an AVV correlator:

$$\begin{split}\partial_\mu^x\,\langle A^\mu V^\alpha V^\beta\rangle = \langle\,(\partial_\mu A^\mu)V^\alpha V^\beta\rangle + \delta(x^0-y^0)\,\langle [A^0, V^\alpha]\,V^\beta\rangle \\ + \delta(x^0-z^0)\,\langle V^\alpha\,[A^0, V^\beta]\rangle = 0\,.\end{split} \tag{4.3.35}$$

The last two terms on the right-hand side are zero because the commutators of the singlet currents vanish, as one can infer from Eq. (3.1.57). The first term produces the pseudoscalar density via the PCAC relation. Repeating this for derivatives with respect to y and z, we arrive at

$$\partial_\mu^x\,\langle A^\mu V^\alpha V^\beta\rangle = 2m\,\langle PV^\alpha V^\beta\rangle\,, \quad \partial_\alpha^y\,\langle A^\mu V^\alpha V^\beta\rangle = 0, \quad \partial_\beta^z\,\langle A^\mu V^\alpha V^\beta\rangle = 0, \tag{4.3.36}$$

without taking into account the anomaly.

The problem is that these diagrams are linearly divergent and therefore not translationally invariant. If one calculates them explicitly to 1-loop order, shifting integration variables by a different momentum routing will produce results that differ by surface terms. The freedom in distributing these surface terms can be used in the regularization procedure when getting rid of all infinite pieces. It turns out that the relations (4.3.36) cannot be satisfied simultaneously, and in order to preserve the vector symmetries the axialvector WTI must pick up the additional anomalous term.

A theorem by **Adler and Bardeen** states that the full structure of the anomaly is already contained in the perturbative one-loop fermion diagrams. Higher-loop corrections do not renormalize the anomaly except for replacing the fields and coupling constants by their renormalized values. For anomaly considerations it is therefore enough to calculate the triangle and rectangle diagrams in Fig. 4.8. These are the superficially divergent ones (in fact, pentagon diagrams should be included as well although they are convergent), and they include an odd number of axial currents and thus an odd number of γ_5 matrices. For example, the anomalous contribution to the η_0 mass in a current correlator arises from quark-disconnected diagrams like the one on the right in Fig. 4.8, which contains intermediate gluon exchanges in the flavor-singlet channel.

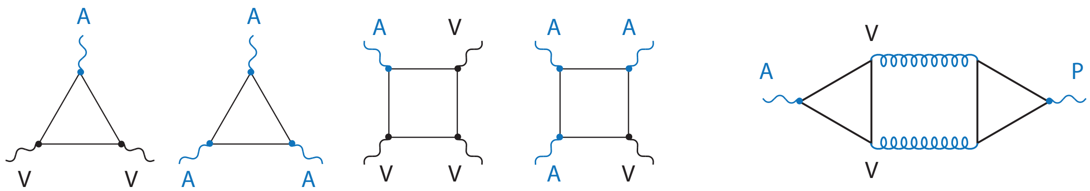

FIG. 4.8: Anomalous 1-loop fermion diagrams.

QED anomaly and $\pi^0 \to \gamma\gamma$ decay. Anomalies have observable consequences. The prime example are the η and η' masses, but in this case the anomalous contribution is also difficult to quantify due to the explicit breaking of chiral symmetry and mixing effects. A much cleaner system is the decay of the π^0 into two photons, which is almost exclusively caused by the axial anomaly from QED effects.

So far we have considered the axial anomaly in QCD (the 'gluon anomaly') which is the relevant one for the $\eta - \eta'$ problem. Quarks couple to gluons, and the quark's flavor-singlet axialvector current A^μ picks up an anomalous term containing the gluonic field-strength tensor. On the other hand, quarks can also couple to photons, which will also produce an anomaly although the related effects are much weaker ($\alpha_{\text{QED}} \ll \alpha_{\text{QCD}}$). If we repeat the derivation for the QED Lagrangian, replace $F^{\mu\nu}$ by the electromagnetic field-strength tensor and the coupling g with e, we obtain the electromagnetic 'photon anomaly' (**Adler-Bell-Jackiw** or **ABJ anomaly**):

$$\partial_\mu A^\mu_a = \frac{e^2}{(4\pi)^2}\,\varepsilon^{\alpha\beta\mu\nu}\,F_{\alpha\beta}\,F_{\mu\nu}\,\mathrm{Tr}_F\left\{\mathsf{t}_a \mathsf{Q}^2\right\}\mathrm{Tr}_C\left\{1\right\}, \quad \mathsf{Q} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.3.37}$$

stated here without the fermion mass term and for $N_f = 3$. The generator t_a comes from the axial transformation and the quark charge matrix Q from the covariant derivative that enters quadratically in the regulator. Since fermions with different flavors have different charges (expressed by Q), photons can also couple to flavor-nonsinglet currents. Therefore, the electromagnetic anomaly produces additional terms for the divergences of the axial currents A^μ and A^μ_a, i.e., for both $U(1)_A$ and $SU(N_f)_A$.

For the $\pi^0 \to \gamma\gamma$ decay, consider the three-point function of an axialvector current and two electromagnetic vector currents:

$$\langle 0|\mathsf{T}\,A^\mu_a(x)\,V^\alpha_{\text{em}}(x_1)\,V^\beta_{\text{em}}(x_2)|0\rangle\,. \tag{4.3.38}$$

The electromagnetic current is proportional to the quark charges and given by

$$V^\mu_{\text{em}}(x) = \bar\psi(x)\,\gamma^\mu\,\mathsf{Q}\,\psi(x) = V^\mu_3(x) + \tfrac{1}{\sqrt{3}}\,V^\mu_8(x)\,. \tag{4.3.39}$$

To lowest order perturbation theory, Eq. (4.3.38) is the AVV triangle diagram in Fig. 4.8 which diverges linearly. However, it also has a spectral representation in terms of pseudoscalar poles, which we can derive in analogy to Eqs. (4.2.34–4.2.36). First, we write down its WTI by acting with the derivative on the index μ:

$$\partial^x_\mu\,\langle 0|\mathsf{T}\,A^\mu_a(x)\,V^\alpha_{\text{em}}\big(\tfrac{z}{2}\big)\,V^\beta_{\text{em}}\big(-\tfrac{z}{2}\big)|0\rangle - 2m\,\langle 0|\mathsf{T}\,P_a(x)\,V^\alpha_{\text{em}}\big(\tfrac{z}{2}\big)\,V^\beta_{\text{em}}\big(-\tfrac{z}{2}\big)|0\rangle = \dots \tag{4.3.40}$$

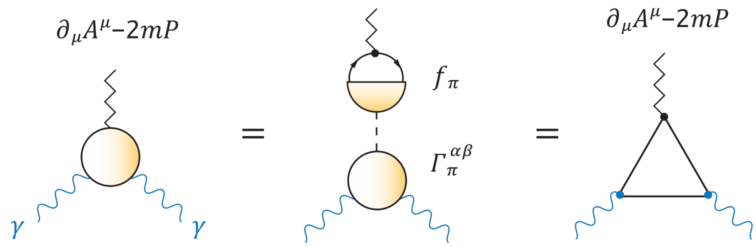

FIG. 4.9: $\pi^0 \to \gamma\gamma$ decay in the chiral limit.

We are interested in the π^0 with $a = 3$; in that case the commutators on the right-hand side obtained from (3.1.74) vanish, because they contain the structure constants $f_{338} = 0$, etc. Instead we have the contribution from the anomaly:

$$\dots = \frac{e^2 D}{(4\pi)^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\langle 0|\mathsf{T}\,F_{\mu\nu}(x)\,F_{\rho\sigma}(x)\,V^\alpha_{\text{em}}\big(\tfrac{z}{2}\big)\,V^\beta_{\text{em}}\big(-\tfrac{z}{2}\big)|0\rangle\,, \tag{4.3.41}$$

where the factor $D = N_c/6$ comes from the flavor and color traces.

If we work out the time orderings on the left-hand side and insert the completeness relation, we can again isolate the Feynman propagator. The pole residues are the two decay constants from Eq. (3.1.142) and the $\pi^0 \to \gamma\gamma$ decay amplitude, defined via

$$\Gamma^{\alpha\beta}_\lambda(z,p) = i\langle\lambda|\mathsf{T}\,V^\alpha_{\text{em}}\big(\tfrac{z}{2}\big)\,V^\beta_{\text{em}}\big(-\tfrac{z}{2}\big)|0\rangle = \int\frac{d^4q}{(2\pi)^4}\,e^{-iqz}\,\Gamma_\lambda(q,p)\,\varepsilon^{\alpha\beta\rho\sigma}q_\rho\,p_\sigma\,. \tag{4.3.42}$$

Its structure in momentum space is due to Lorentz and parity invariance: p is the pion momentum, q the relative momentum between the photons, and the only possible Lorentz tensor is $\varepsilon^{\alpha\beta\rho\sigma}q_\rho\,p_\sigma$. Integrating (4.3.40) over x and z, the poles drop out again and the analogue of Eq. (4.2.36) becomes

$$\lim_{\substack{p\to 0\\ q\to 0}}\sum_\lambda f_\lambda\,\Gamma^{\alpha\beta}_\lambda(q,p) = \lim_{\substack{p\to 0\\ q\to 0}} f_\pi\,\Gamma^{\alpha\beta}_\pi(q,p) = 0\,, \tag{4.3.43}$$

as long as we discard the anomaly on the right-hand side. We have again removed the sum over λ because the decay constants are zero for all excited states with $m_\lambda \neq 0$. Since the transition matrix elements are defined at $p^2 = m_\pi^2 = 0$, this is a chiral-limit relation. Hence, the decay amplitude should be zero, which is known as the **Sutherland-Veltman theorem**.

In order to take the anomaly into account, we would have to work out the right-hand side of Eq. (4.3.41). However, since the anomaly is already produced in the lowest order perturbation theory, it is sufficient to start again from Eq. (4.3.40) and work out its perturbative 1-loop contributions, the AVV and PVV triangle diagrams. The ambiguity in shifting integration variables produces just the anomalous term. The result has the same structure in momentum space $\sim \varepsilon^{\alpha\beta\rho\sigma}q_\rho\,p_\sigma$, and the resulting decay amplitude becomes $\Gamma_\pi(0,0) = e^2 D/(2\pi^2 f_\pi)$. The calculated $\pi \to \gamma\gamma$ decay width using this result is 7.862 eV; the experimental value is 7.8±0.9 eV. Therefore, the neutral pion decay does not probe the nonperturbative structure of QCD at all — it is completely determined by the axial anomaly.