2.2 Quantization of QCD

So far we have discussed the Lagrangian and action of QCD, Eq. (2.1.29), which define the classical field theory. What are the consequences of going to the quantum field theory (QFT)? There are two standard methods to transform the classical action into a QFT. One is the canonical formalism, where the fields are treated as operators on a Fock space and canonical (anti-)commutation relations are imposed. The other is the path-integral formalism where an integral over all fields is performed, which provides an intuitive picture of how quantum corrections augment the classical field configurations. Both methods are equivalent and we will use them in combination, depending on what better suits our needs.

In the following we briefly recall some basic concepts of QFT. For illustration, we work in a generic theory with one species of fields, $\phi(x)$, defined by the classical action $S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$; in QCD, ϕ would stand for the fields $\phi \in \{\psi, \bar{\psi}, A^\mu_a\}$.

2.2.1 Canonical quantization

Assuming that you have heard a standard QFT course and went through all the machinery, how would you summarize the basic ideas in a few words? Fortunately, the formalism behind a QFT requires only a small number of axioms:

Relativity and unitarity. In a QFT, the field $\phi(x)$ is interpreted as an operator on a Fock space with elements $|p_1 \dots p_n\rangle$, $n = 0, 1, 2, \dots$ This includes the vacuum $|0\rangle$, single-particle states $|p\rangle$ with four-momentum p, as well as multi-particle states. These states transform under *unitary* representations $U(\Lambda, a)$ of the Poincaré group, which provide a probability interpretation for S-matrix elements $\langle p_1 \dots p_n | q_1 \dots q_r \rangle$.

More details on the Poincaré group can be found in Appendix B. The group consists of translations (with group element a^μ), rotations and boosts (which define the Lorentz group with group element Λ). Because of the boosts, the Lorentz group is not compact and therefore its finite-dimensional representations are not unitary. However, the classical fields transform just under such finite-dimensional representations: a scalar field $\phi'(x') = \phi(x)$ is invariant, a Dirac field $\psi(x)$ transforms under a four-dimensional spinor representation $\psi'(x') = D(\Lambda)\,\psi(x)$, a vector field $A^\mu(x)$ under a four-dimensional vector representation $A'^\mu(x') = \Lambda^\mu{}_\nu A_\nu(x)$, etc. Because these representations are not unitary, there is no hope for extracting probability amplitudes from the classical fields. *Unitary* representations are *infinite-dimensional*, which is why we need an infinite-dimensional Fock space. In other words, implementing relativity (which enters through the boosts) *while* maintaining unitarity takes us from quantum mechanics to QFT, where we interpret the fields as operators on the Fock space — instead of fields like in the classical field theory or wave functions like in quantum mechanics.

Causality. The second basic requirement is that two measurements with a spacelike distance cannot affect each other:

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \qquad \text{for} \qquad (x-y)^2 < 0\,. \tag{2.2.1}$$

Here, $\mathcal{O}_1$ and $\mathcal{O}_2$ are observables such as quark bilinears $\bar{\psi}\,\Gamma\,\psi$, where Γ is some Dirac matrix. For bosonic fields, this requirement is implemented by imposing spacelike commutation relations $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$, whereas for fermionic fields we need anticommutation relations $\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = 0$. The equal-time commutation or anticommutation relations are special cases of those.

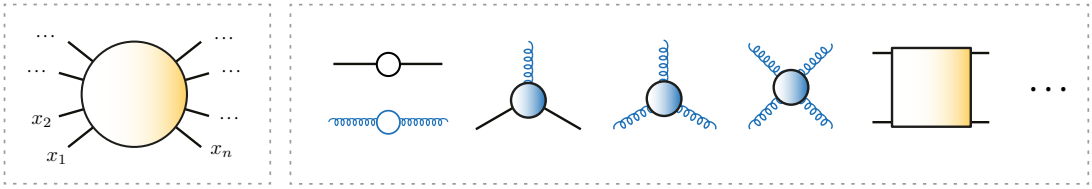


FIG. 2.2: Generic correlation function (left) and some of the elementary correlation functions in QCD (right)

These are some of the pillars on which a QFT is built and they summarize the first part of a typical QFT course: one develops the formalism for free scalar, spinor and vector fields, based on the representations of the Lorentz group, and studies the implementation of symmetries through the Noether theorem. Below we will discuss another pillar, the spectral condition, and depending on what we are after we could also add renormalizability or (for gauge theories) local gauge invariance to the list.

Interacting QFT. Unfortunately, after switching on interactions, one quickly runs into trouble at the operator level (cf. Haag's theorem: a free field remains always free). To avoid these problems, the hard statements that can be made in an interacting QFT are those for *matrix elements* of operators, which therefore become the central objects of interest. The **correlation functions**, also called n-point functions or Green functions, are the vacuum expectation values of time-ordered products of fields:

$$G(x_1, \dots x_n) := \langle 0 | \mathsf{T}\, \phi(x_1) \dots \phi(x_n) | 0 \rangle \,. \tag{2.2.2}$$

Pictorially, these are blobs with n legs, one for each spacetime point $x_1 \dots x_n$, which contain all possible interactions between the particles that can happen inside (Fig. 2.2). The simplest example is a two-point function, a **propagator**, which contains the (self-) interactions of a single particle and which for free fields becomes the usual Feynman propagator. For theories with different types of fields there can be interactions between different particles, and Fig. 2.2 shows some of the correlation functions in QCD: the quark and gluon propagators and some of their three- and four-point functions.

Why are these correlation functions relevant? For one, the LSZ formula tells us that they are related to S-matrix elements:

$$\begin{aligned} &\mathrm{FT}\left[G(x_1 \dots x_n, y_1 \dots y_r)\right] \\ &\quad = \left(\prod_{i=1}^{n} \frac{i\sqrt{Z}}{p_i^2 - m_i^2 + i\epsilon} \right) \left(\prod_{j=1}^{r} \frac{i\sqrt{Z}}{q_j^2 - m_j^2 + i\epsilon} \right) \langle p_1 \dots p_n, \mathrm{out} \,|\, q_1 \dots q_r, \mathrm{in} \rangle_{\mathrm{conn.}} \,, \end{aligned} \tag{2.2.3}$$

That is, we can extract the connected S-matrix element for an $n \to r$ scattering process (the invariant amplitude $\mathcal{M}$) if we calculate the respective correlation function in momentum space ('FT' stands for Fourier transform), go to the kinematic limit where the external propagators are onshell, and amputate those external propagators. If the particles carry spinor or vector quantum numbers, we must also contract with onshell spinors or polarization vectors. Once we know $\mathcal{M}$, we can compute the cross section from $|\mathcal{M}|^2$ and compare it to experiment.

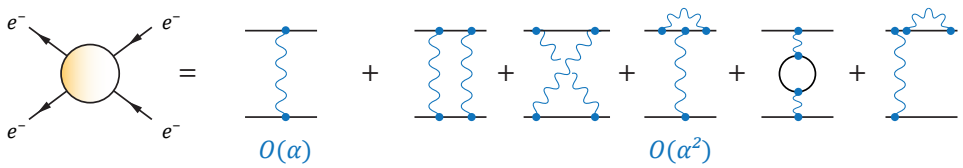


FIG. 2.3: Lowest-order perturbative diagrams that contribute to Møller scattering in QED. Topologies like those in the last diagram do not survive the amputation.

This is the standard recipe for a typical QFT calculation. The practical problem is of course that we first need to know *how* to calculate the correlation functions in question. If the coupling constant of the theory is small at the momentum scales of interest, we can use **perturbation theory** and expand them into Feynman diagrams, where we neglect diagrams above a certain loop order that are suppressed by the small coupling constant in front. (The other practical issue is renormalization but this is well understood, see Sec. 2.3.)

This recipe has turned out to be extremely successful, with the prime example given by **QED**. In that case, the dimensionless coupling $\alpha_{\rm QED} = e^2/(4\pi)$ is indeed small; inserting dimensions, we have[1]

$$\alpha_{\rm QED} = \frac{e^2}{4\pi}\,\frac{1}{\hbar c \varepsilon_0} \approx \frac{1}{137}\,. \tag{2.2.4}$$

If we perform a loop expansion for a given correlation function, then higher loop diagrams come with higher powers of the coupling and in practice it may even be sufficient to stick with the lowest (tree-level) order. For example, the electron four-point function in Fig. 2.3 describes both the Møller ($e^-e^- \to e^-e^-$) and Bhabha ($e^+e^- \to e^+e^-$) scattering processes. The leading contribution to Møller scattering is the one-photon exchange diagram, which leads to the Mott cross section plus spin terms (more on that later). The smallness of $\alpha_{\rm QED}$ has contributed to the successes of QED, where many observables can be calculated quite precisely by going to higher orders in perturbation theory. This has led to a variety of precision measurements of $\alpha_{\rm QED}$, from the anomalous magnetic moment ('$g-2$') of electrons and muons, measurements of the Rydberg constant, the energy level splittings in atoms, etc.

Unfortunately, when we try to apply the same principles to QCD we are confronted with two challenges that complicate matters enormously. The first difficulty is that the coupling $\alpha_{\rm QCD}$ becomes large at low momenta (see Section 2.3) and invalidates a perturbative expansion. Unfortunately this is just the region that is relevant for hadron physics, so we must look for **nonperturbative methods** to calculate these correlation functions at low momenta. The second difficulty is more fundamental: confinement entails that it is pointless to calculate invariant scattering amplitudes of quarks and gluons because we can never measure such processes. What we can measure are reactions between hadrons (e.g. NN or $N\pi$ scattering), or hadrons that interact with leptons through the electroweak interaction (e^+e^- annihilation, eN scattering etc.). We will return to this point in Section 3.1.3, where we show how one can still extract measurable information from QCD's elementary correlation functions.

[1] In natural units $\hbar = 1$ sets the units of action $[ML^2/T]$, $c = 1$ sets the units of velocity $[L/T]$ and $\hbar c\varepsilon_0 = 1$ the units of charge $[C^2]$.

In any case, what still stands is that the correlation functions in Eq. (2.2.2) encode the full content of the QFT, namely through the quantum effective action (more on this in Sec. 2.2.2). Thus, if we knew all of them — and there are infinitely many — it would be equivalent to having *solved* the QFT. For this reason, QCD's correlation functions will be the central quantities of interest throughout this course.

Spectral representation. Returning to the remaining basic property of QFT that we passed over before, let us assume that each Fock state, which we generically denote by $|\lambda\rangle$, is an eigenstate of the Hamiltonian with definite energy and momentum. The vacuum $|0\rangle$ has vanishing energy and momentum. A one-particle state $|\boldsymbol{p}\rangle$ has momentum $\boldsymbol{p}$ and energy $E_p = \sqrt{\boldsymbol{p}^2 + m^2}$. Multiparticle states, which are characterized by a center-of-mass momentum $\boldsymbol{p}$ and relative momenta among the particles, form a continuum: For example, the lowest possible energy of a two-particle state in its rest frame ($\boldsymbol{p} = 0$) is $2m$, but since the two particles can have relative momentum, which contributes to their total energy, the state can have *any* energy $E_p(\lambda) = (\boldsymbol{p}^2 + m_\lambda^2)^{1/2}$, where $m_\lambda \geq 2m$ is the invariant mass of the state (its energy in the rest frame). The eigenvalue spectrum of H then has the form shown in Fig. 2.4: a one-particle state sits on its mass shell, a two-particle state forms a continuum with $m_\lambda \geq 2m$ and so on.

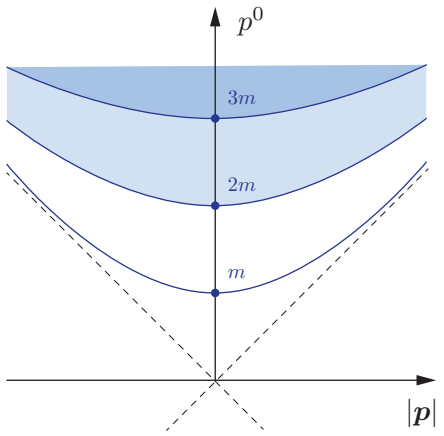


FIG. 2.4: Eigenvalue spectrum of the Hamiltonian in terms of one-particle states with mass m and multiparticle states with invariant mass $m_\lambda \geq 2m$.

Based on this, we write the completeness relation of the Fock space as

$$\mathbb{1} = \sum_\lambda \frac{1}{(2\pi)^3} \int d^4p\, \Theta(p^0)\, \delta(p^2 - m_\lambda^2)\, |\lambda\rangle\langle\lambda| = \sum_\lambda \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p}\, |\lambda\rangle\langle\lambda|\,, \tag{2.2.5}$$

where the sum over λ is formal and includes integrals over relative momenta. The Lorentz-invariant integral measure ensures that we only integrate over $p^2 = m_\lambda^2 \Leftrightarrow p_0^2 = E_p^2$, and we used

$$\delta(p^2 - m_\lambda^2) = \delta(p_0^2 - E_p^2) = \frac{\delta(p_0 - E_p) + \delta(p_0 + E_p)}{2E_p}\,, \tag{2.2.6}$$

where only the first term survives because the step function enforces $p_0 > 0$.

It is easy to show that if we insert the completeness relation into the propagator $G(x,y) = \langle 0|\mathsf{T}\,\phi(x)\,\phi(y)|0\rangle$, we arrive at the **spectral representation** for the propagator in momentum space:

$$G(p^2) = \sum_\lambda \frac{iR_\lambda}{p^2 - m_\lambda^2 + i\epsilon} = \int_0^\infty \frac{ds}{2\pi} \frac{i\rho(s)}{p^2 - s + i\epsilon}\,. \tag{2.2.7}$$

Here we defined the **spectral function**

$$\rho(s) = \sum_\lambda 2\pi\delta(s - m_\lambda^2)\, R_\lambda\,, \tag{2.2.8}$$

where the sum over λ is again formal and denotes relative-momentum integrations.

The proof goes as follows: By inserting the completeness relation, we obtain

$$\begin{aligned} G(x,y) &= \langle 0|\mathsf{T}\,\phi(x)\,\phi(y)|0\rangle \\ &= \Theta(x^0-y^0)\,\langle 0|\phi(x)\,\phi(y)|0\rangle + \Theta(y^0-x^0)\,\langle 0|\phi(y)\,\phi(x)|0\rangle \\ &= \sum_\lambda \frac{1}{(2\pi)^3}\int\frac{d^3p}{2E_p}\left\{\begin{array}{r}\Theta(x^0-y^0)\,\langle 0|\phi(x)|\lambda\rangle\,\langle\lambda|\phi(y)|0\rangle \\ +\Theta(y^0-x^0)\,\langle 0|\phi(y)|\lambda\rangle\,\langle\lambda|\phi(x)|0\rangle\end{array}\right\}. \end{aligned} \tag{2.2.9}$$

Now we use the transformation properties of the operator $\phi(x)$ under Poincaré transformations $U(\Lambda,a)$ and in particular translations $U(1,a)$:

$$U(\Lambda,a)\,\phi(x)\,U(\Lambda,a)^{-1} = \phi(\Lambda x + a) \qquad \Rightarrow \qquad U(1,a)\,\phi(0)\,U(1,a)^{-1} = \phi(a)\,. \tag{2.2.10}$$

On the other hand, applying a translation to a Fock state only produces a phase, whereas the vacuum remains invariant:

$$U(1,a)\,|\lambda(p)\rangle = e^{ip\cdot a}\,|\lambda(p)\rangle\,, \qquad U(1,a)\,|0\rangle = |0\rangle\,. \tag{2.2.11}$$

In combination, we have

$$\begin{aligned} \langle 0|\phi(x)|\lambda\rangle &= \langle 0|U(1,x)\,\phi(0)\,U(1,x)^{-1}|\lambda\rangle = \langle 0|\phi(0)|\lambda\rangle\, e^{-ip\cdot x}\,, \\ \langle\lambda|\phi(x)|0\rangle &= \langle 0|\phi(0)|\lambda\rangle^*\, e^{ip\cdot x}\,. \end{aligned} \tag{2.2.12}$$

The remaining matrix element $\langle 0|\phi(0)|\lambda(p)\rangle$ is Lorentz-invariant, so for a single-particle state it can only depend on $p^2 = m^2$ which is just a number (for multiparticle states it still depends on the relative momenta). Denoting $|\langle 0|\phi(0)|\lambda\rangle|^2 = R_\lambda$ and $z = x - y$, we arrive at

$$G(z) = \sum_\lambda R_\lambda \int\frac{d^3p}{2E_p}\left[\frac{\Theta(z^0)\,e^{-ip\cdot z} + \Theta(-z^0)\,e^{ip\cdot z}}{(2\pi)^3}\right]_{p^0=E_p(\lambda)} = \sum_\lambda R_\lambda\, D_F(z,m_\lambda). \tag{2.2.13}$$

The integral is nothing but the Feynman propagator

$$D_F(z,m_\lambda) = \int\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot z}\,\frac{i}{p^2 - m_\lambda^2 + i\epsilon}\,, \tag{2.2.14}$$

and we arrive at the result

$$G(z) = \int\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot z}\sum_\lambda\frac{iR_\lambda}{p^2 - m_\lambda^2 + i\epsilon} \tag{2.2.15}$$

from where we can read off the propagator $G(p^2)$ in momentum space.

Note that Eq. (2.2.7) holds for the full ('dressed') propagator of the theory, i.e., knowledge of the spectral function is equivalent to the knowledge of the full two-point function. The spectral function is related to the analytic structure of the propagator. For a typical theory it is positive and has the form of Fig. 2.5: The one-particle states lead to an isolated $\delta-$function peak at $s = m^2$, which defines the mass m of the particle in the full theory (which is not the mass parameter in the Lagrangian!) from the lowest-lying pole location of the propagator in momentum space. The continuum of $n-$particle states begins at $s \geq (2m)^2$, which leads to a branch cut in the propagator starting at $p^2 = 4m^2$.

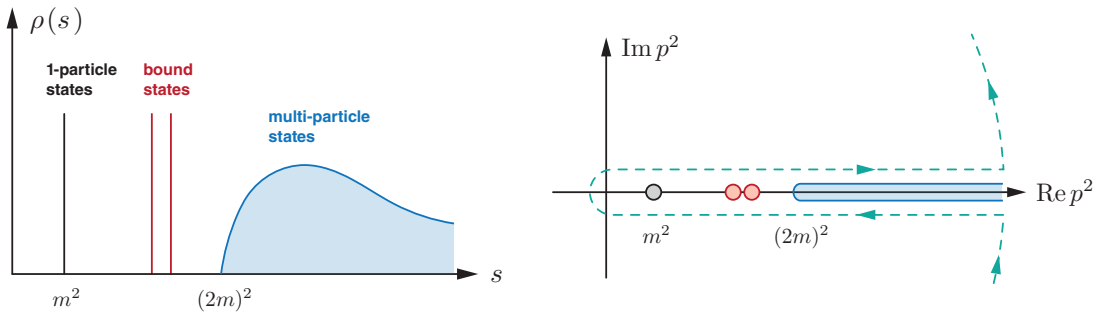


FIG. 2.5: Left: spectral function of a typical field theory, with a single-particle peak at $s = m^2$ and a multiparticle continuum for $s \geq 4m^2$, together with possible bound states. Right: Analytic structure of the corresponding propagator with a single-particle pole, bound-state poles and a branch cut above $p^2 = 4m^2$. The dashed line is the contour integration path that encloses the domain of analyticity.

To see this, define $G_E(w) = iG(w)$ with $w = u + iv \in \mathbb{C}$ and $u, v \in \mathbb{R}$. This function (which is the *Euclidean* propagator) satisfies the Schwartz reflection principle $G_E^*(w) = G_E(w^*)$ for analytic functions. We can use the Cauchy formula to determine $G_E(w)$ for w inside its domain of analyticity:

$$G_E(w) = \frac{1}{2\pi i} \oint ds\, \frac{G_E(s)}{s - w}\,. \tag{2.2.16}$$

If $G_E(w)$ has only singularities on the positive real axis, we can choose the integration path in Fig. 2.5, and if it falls off sufficiently fast for $|w| \to \infty$, what remains is the integral

$$\begin{aligned} G_E(w) &= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_0^\infty ds \left[\frac{G_E(s + i\epsilon)}{s - w + i\epsilon} - \frac{G_E(s - i\epsilon)}{s - w - i\epsilon} \right] \\ &= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_0^\infty ds \left[\frac{G_E(s + i\epsilon)}{s - u - i(v - \epsilon)} - \frac{G_E^*(s + i\epsilon)}{s - u - i(v + \epsilon)} \right]. \end{aligned} \tag{2.2.17}$$

For $|v| > \epsilon$, w always lies in the domain of analyticity and we can take the limit $\epsilon \to 0$ to obtain

$$G_E(w) = \frac{1}{\pi} \int_0^\infty ds \frac{\mathrm{Im}\, G_E(s)}{s - u - iv}\,. \tag{2.2.18}$$

This is the same formula as Eq. (2.2.7) if we identify $u = p^2$ and $v = \epsilon$, and therefore

$$\rho(s) = 2\, \mathrm{Im}\, G_E(s)\,. \tag{2.2.19}$$

Thus, if the spectral function is non-zero for $s > 4m^2$, the imaginary part of the propagator is discontinuous above that threshold — it has a branch cut.

In this way, single-particle asymptotic states produce poles in the propagator and multi-particle states produce cuts. In fact, every new multi-particle production threshold opens up another branch cut and thus another Riemann sheet, so the propagator becomes a multi-valued function. Vice versa, the existence of a spectral representation implies that the propagator has only singularities on the timelike axis $p^2 > 0$ but not in the complex plane of the first Riemann sheet: had there been singularities inside the integration contour in Eq. (2.2.16), their residues would produce further terms in (2.2.18) and the identification with Eq. (2.2.7) would no longer go through.

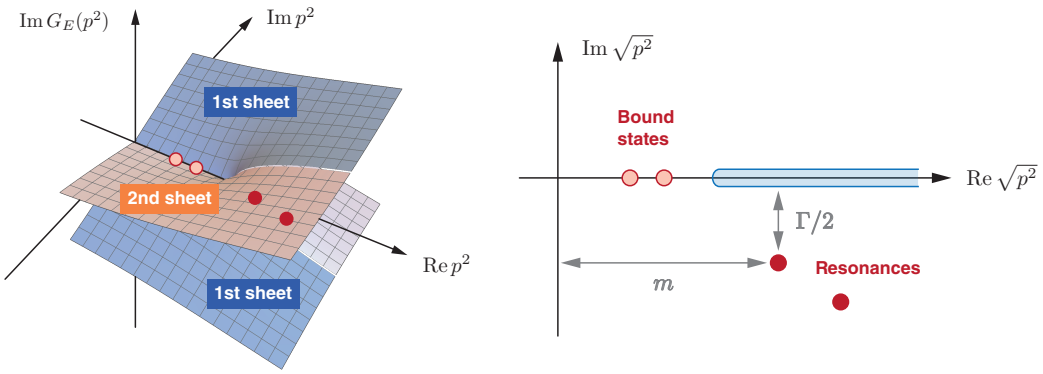


FIG. 2.6: Bound states have poles on the positive real axis below the threshold, whereas resonances have poles on the second Riemann sheet. (The left figure is only meant for illustration, since what is shown is really just the square-root function.)

At this point there is no consensus to what extent a spectral representation can be formulated for gauge theories and/or theories with confinement. For the scalar example the situation is clear: the operator $\phi(x)$ creates single- or multiparticle states $|\lambda\rangle$ with definite mass m, momentum $\boldsymbol{p}$ and energy E_p. But what about QCD? Certainly, we can count pions $|\pi\rangle$, nucleons $|N\rangle$ and other stable states as asymptotic states in QCD — but these are all *bound states*. The corresponding multiparticle states are $|\pi\pi\rangle$, $|\pi\pi\pi\rangle$, $|NN\rangle$, $|N\bar{N}\rangle$ etc. Indeed, the sensible matrix elements in QCD are all of the form

$$\langle 0|\dots|0\rangle, \quad \langle 0|\dots|\pi\rangle, \quad \langle N|\dots|N\rangle, \quad \langle 0|\dots|\pi\pi\rangle, \quad \text{etc.}, \tag{2.2.20}$$

where the dots denote time-ordered products of quark and gluon field operators, and the initial and final states contain either the vacuum or stable particles. But what about quarks and gluons? What is the 'mass' or 'energy' of a quark or gluon if we cannot measure it? Should we even count them as Fock states in the sense of Eq. (2.2.5)? If we didn't, we would break the link between the 'masses' of quarks and gluons and the poles in their propagators, and there would be no reason why QCD's elementary correlation functions *should* have singularities on the positive real axis only — they could also lie in the complex plane, as long as their effects cancel out in observable scattering amplitudes like those in Eq. (2.2.20). (It is often said that *causality* alone restricts the singularities to the positive real axis, but the proof of this statement also assumes a spectral condition.) It may still be possible to formulate *generalized* spectral representations also for quarks and gluons, which restrict their singularities again to the positive real axis. In any case, in practice from now on we assume that $|\lambda\rangle$ only refers to asymptotic states with a well-defined mass and energy, which in QCD are bound states.

Resonances. There is another way how poles can move away from the real axis, even in the presence of a spectral representation: A bound-state pole can pass a multiparticle production threshold and become a resonance, i.e., an unstable state. By doing so, it acquires a width, which means an imaginary part. This does not contradict our earlier statements, since the spectral representation still implies that the first ('physical') sheet must be free of singularities. As a consequence, resonances can only appear as poles on 'unphysical' higher Riemann sheets.

The situation is sketched in Fig. 2.6. The resonance pole location is usually written as $p^2 = (m - i\Gamma/2)^2$, so that in the complex $\sqrt{p^2}$ plane the real part is quoted as the mass of the resonance and the imaginary part as half the width. The width is related to the inverse lifetime of the particle: a bound state has an infinite lifetime whereas very short-lived resonances have poles far in the complex plane.

FIG. 2.7: Integration paths according to the $i\epsilon$ prescription.

Where does $i\epsilon$ come from? As a final remark, what is the origin of the '$i\epsilon$ prescription' that shows up in formulas like (2.2.3) and (2.2.14)? You probably first encountered it when taking the Fourier transform of the propagator $i/(p^2 - m^2)$ in the free field theory. After splitting the d^4p integral into d^3p and dp_0, this function has poles at $p_0 = \pm E_p = \pm\sqrt{\boldsymbol{p}^2 + m^2}$, which depend on $|\boldsymbol{p}|$ as indicated in the left of Fig. 2.7. To reproduce the Feynman propagator $\langle 0|\mathsf{T}\,\phi(x)\,\phi(y)|0\rangle$, one must integrate slightly below the p_0 axis for $\mathrm{Re}\,p_0 < 0$ and slightly above for $\mathrm{Re}\,p_0 > 0$.

In general, the $i\epsilon$ prescription follows from the imaginary-time boundary conditions when projecting correlation functions onto the interacting vacuum $|0\rangle$:

$$\sum_{n=0}^{\infty} e^{-iE_nT}|n_{\rm f}\rangle\langle n_{\rm f}|0\rangle \xrightarrow{T\to\infty(1-i\epsilon)} e^{-iE_0T}|0_{\rm f}\rangle\langle 0_{\rm f}|0\rangle\,, \tag{2.2.21}$$

which removes the higher-energy contributions of the free-particle states $|n_{\rm f}\rangle$. This is equivalent to imposing boundary conditions for every d^4x integral (such as the one in the action of the theory) and every d^4p integral,

$$\int d^4x = \int d^3x \int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dx_0 \qquad \Leftrightarrow \qquad \int d^4p = \int d^3p \int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dp_0\,, \tag{2.2.22}$$

or, alternatively, shifting the propagator poles in the p_0 variable by $i\epsilon$.

In practice it is convenient to employ a **Euclidean metric** by defining $x_4 = ix_0$ with $x_4 \in \mathbb{R}$ and formulate the theory directly in Euclidean spacetime. In that case boundary conditions become irrelevant and the weight factor $e^{-S_E[\phi]}$ in the path integral defines a probability measure. This is what is usually done in practical calculations using path integrals (here we will stick to the Minkowski metric, though). For momentum-space integrals it corresponds to a **Wick rotation**, where the integration path in Fig. 2.7 does not proceed from left to right but from bottom to top.

Strictly speaking, for any finite ϵ, Eq. (2.2.22) implies to integrate not *slightly* below and above the p_0 axis, but to start at the bottom left corner and integrate to the top right. This does not make any difference for the Fourier transform of the propagator, but it is useful to remember when calculating loop integrals, whose pole locations can line up like in the right of Fig. 2.7 for certain kinematical situations. If one integrates over d^3p first, the poles produce branch cuts in the complex p_0 plane, and to avoid them one has to deform the integration contour accordingly — both with or without a Wick rotation. If one integrates over p_0 first, one must pick the correct residues. If this is done properly, results in Minkowski and Euclidean space are the same.

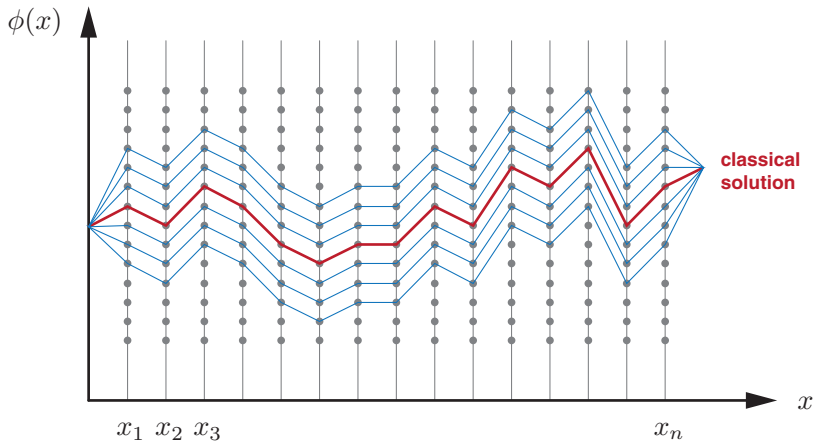


FIG. 2.8: Visualization of a path integral. The integration over all values of $\phi(x_i)$ at each spacetime point x_i is equivalent to an integration over all possible field configuration $\phi(x)$.

2.2.2 Path-integral quantization

Path integrals in QFT. Another way to quantize a field theory is the path-integral formalism. The central object is the path integral

$$\int \mathcal{D}\phi\, e^{iS[\phi]}\,(\dots) = \lim_{n\to\infty} \int d\phi(x_1)\dots d\phi(x_n)\, e^{iS[\phi]}\,(\dots)\,, \tag{2.2.23}$$

which can be pictorially understood as in Fig. 2.8: at each spacetime point x_i we integrate over all possible values $\phi(x_i)$, which amounts to an integration over all *paths* in the field space, i.e., all possible field configurations. If we had different types of fields with additional group representation labels or Lorentz-Dirac indices, the product would go over all of them as well. With $e^{iS[\phi]}$ as probability measure (for this one needs the $i\epsilon$ prescription or go to Euclidean space), the integral picks up the quantum corrections to the classical path defined by the classical equations of motion $\delta S[\phi]/\delta\phi = 0$. As long as the system is sufficiently 'classical', the classical solutions will dominate the result; otherwise many trajectories will contribute.

One can show that the correlation functions (2.2.2) in the path-integral formalism are given by

$$G(x_1,\dots x_n) = \langle 0|\mathsf{T}\,\phi(x_1)\dots\phi(x_n)|0\rangle = \frac{\int \mathcal{D}\phi\, e^{iS[\phi]}\,\phi(x_1)\dots\phi(x_n)}{\int \mathcal{D}\phi\, e^{iS[\phi]}}\,. \tag{2.2.24}$$

Even though our notation does not distinguish it, one should remember that the $\phi(x_i)$ in the vacuum expectation value are *operators* acting on the state space, whereas the $\phi(x_i)$ in the path integral are merely *functions* but not operators. For fermionic fields, they are anticommuting Grassmann fields to implement the anticommutativity (see Sec. 2.2.3). To avoid the cumbersome distinction between operators and functions, and to reflect the statistical nature of the path integral as a quantum expectation value, it is common to denote the correlation functions by $G(x_1,\dots x_n) = \langle\phi(x_1)\dots\phi(x_n)\rangle$.

The apparent drawback of the path-integral approach is that it is quite *hard* to actually calculate path integrals. With the exception of a few simple cases, such calculations usually have to be performed numerically. This is done in **lattice QFT**, where spacetime is discretized and path integrals are calculated by statistical Monte-Carlo sampling. This is the most direct way to compute correlation functions and the hadronic observables they encode from QCD, and over the last decades lattice QCD has made spectacular progress in that arena.

Functional derivatives. There is another way of making the path-integral approach useful without *actually* calculating path integrals. Namely, one can generate the correlation functions from the **partition function** $Z[J]$ by adding a source term with an external source $J(x)$:

$$Z[J] = \int \mathcal{D}\phi \, e^{i\left(S[\phi] - \int d^4x \, \phi(x) J(x)\right)} \,. \tag{2.2.25}$$

If we take a functional derivative of the source term, we obtain

$$\frac{i\delta}{\delta J(x_1)} \left[-i \int d^4x \, \phi(x) J(x) \right] = \phi(x_1) \,. \tag{2.2.26}$$

Taking the functional derivative of the partition function then yields

$$\begin{aligned} \frac{i\delta}{\delta J(x_1)} Z[J] &= \int \mathcal{D}\phi \, e^{i(\dots)} \, \phi(x_1) \,, \\ \frac{i\delta}{\delta J(x_1)} \frac{i\delta}{\delta J(x_2)} Z[J] &= \int \mathcal{D}\phi \, e^{i(\dots)} \, \phi(x_1) \, \phi(x_2) \end{aligned} \tag{2.2.27}$$

etc., which can be generalized to an arbitrary polynomial function(al) of the fields:

$$f\left[\frac{i\delta}{\delta J}\right] Z[J] = \int \mathcal{D}\phi \, e^{i(\dots)} \, f[\phi] \,. \tag{2.2.28}$$

Comparing this with Eq. (2.2.24), we see that we can generate the correlation functions by an n-fold derivative of $Z[J]$, dividing by $Z[0]$ and finally letting $J \to 0$:

$$G(x_1 \dots x_n) = \frac{i\delta}{\delta J(x_1)} \dots \frac{i\delta}{\delta J(x_n)} \bigg|_{J=0} \frac{Z[J]}{Z[0]} \,. \tag{2.2.29}$$

The two-fold functional derivative of $Z[J]$ is the two-point function, the three-fold derivative the three-point function, etc. Vice versa, the partition function can be written as

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \, e^{iS[\phi]} \sum_{n=0}^{\infty} \frac{(-i)^n}{n} \left(\int d^4x \, J(x) \, \phi(x) \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n} \int d^4x_1 \dots d^4x_n \, J(x_1) \dots J(x_n) \underbrace{\int \mathcal{D}\phi \, e^{iS[\phi]} \, \phi(x_1) \dots \phi(x_n)}_{Z[0] \, G(x_1, \dots x_n)} \,. \end{aligned} \tag{2.2.30}$$

Thus, we can generate all correlation functions from $Z[J]$, and $Z[J]$ can be reconstructed from the knowledge of all correlation functions. They contain the same information as the partition function, which defines the QFT.

In the following it will be convenient to leave the $J-$dependence intact, at least for intermediate steps in calculations, so that the 'physics' is recovered in the end when setting $J = 0$. Through Eq. (2.2.28), we define correlation functions in the presence of the source J as

$$\langle f[\phi]\rangle_J := \frac{\int \mathcal{D}\phi\, e^{i\left(S[\phi]-\int d^4x\,\phi(x)J(x)\right)} f[\phi]}{\int \mathcal{D}\phi\, e^{i\left(S[\phi]-\int d^4x\,\phi(x)J(x)\right)}} = \frac{1}{Z[J]}\, f\left[\frac{i\delta}{\delta J}\right] Z[J]\,. \tag{2.2.31}$$

Perturbation theory. The path-integral approach is very useful for doing calculations in perturbation theory. To do so, we split the action into a non-interacting and an interacting part:

$$S[\phi] = S_0[\phi] + g\, S_I[\phi]\,. \tag{2.2.32}$$

Then we can write

$$e^{ig\, S_I[\phi]}\, e^{-i\int d^4x\,\phi(x)J(x)} = e^{ig\, S_I\left[\frac{i\delta}{\delta J}\right]}\, e^{-i\int d^4x\,\phi(x)J(x)}\,, \tag{2.2.33}$$

since for a small coupling constant g we can expand $e^{ig\, S_I[\phi]}$ in powers of g and each term consists of polynomials of the fields. In this way we can pull out the interacting part of the exponential from the path integral,

$$Z[J] = e^{ig\, S_I\left[\frac{i\delta}{\delta J}\right]} \int \mathcal{D}\phi\; e^{i\left(S_0[\phi]-\int d^4x\,\phi(x)J(x)\right)} = \sum_n \frac{(ig)^n}{n!} \left(S_I\left[\frac{i\delta}{\delta J}\right]\right)^n Z_0[J]\,, \tag{2.2.34}$$

where the remaining path integral $Z_0[J]$ is calculable in a closed form since it only depends on the free action. For example, in the scalar theory one obtains

$$Z_0[J] = \int \mathcal{D}\phi\; e^{i\left(S_0[\phi]-\int d^4x\,\phi(x)J(x)\right)} = Z_0[0]\, e^{-\frac{1}{2}\int d^4x\, J(x)\, D_F(x,y)\, J(y)}\,, \tag{2.2.35}$$

where the constant $Z_0[0]$ absorbs the remaining path integral and $D_F(x,y)$ is the Feynman propagator. In this way, $Z[J]$ can be computed order by order in perturbation theory: The result for $n = 0$ is the free theory, $n = 1$ gives the $\mathcal{O}(g)$ correction, and so on. The correlation functions (2.2.29) follow from functional derivatives, where the constant $Z_0[0]$ drops out.

Quantum equations of motion. While the machinery of perturbation theory is equally straightforward to set up using canonical quantization, the power of the path-integral approach lies in its non-perturbative applications. This leads to the so-called **functional methods**, where relations between the correlation functions of the theory are derived in the form of integral or differential equations (or both).

The **Dyson-Schwinger equations (DSEs)** are the quantum equations of motion of a QFT and can be derived with almost no assumptions. The partition function $Z[J]$ is invariant under a shift $\phi'(x) = \phi(x) + \epsilon(x)$, since this is just a relabeling of the fields under the integral, so we can write

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi'\, e^{i\left(S[\phi']-\int d^4x\,\phi'(x)J(x)\right)} \\ &= \int \mathcal{D}\phi\; e^{i\left(S[\phi]-\int d^4x\,\phi(x)J(x)\right)+i\int d^4x\,\epsilon(x)\left(\frac{\delta S[\phi]}{\delta\phi(x)}-J(x)\right)} \\ &= Z[J]\left\langle e^{i\int d^4x\,\epsilon(x)\left(\frac{\delta S[\phi]}{\delta\phi(x)}-J(x)\right)}\right\rangle_J . \end{aligned} \tag{2.2.36}$$

FIG. 2.9: Full, connected and 1PI diagrams in the four-point function of ϕ^4 theory (permutations not shown).

In the second line we assumed that the path integral measure $\mathcal{D}\phi$ is also invariant, and in the third line we inserted the definition (2.2.31). Since $\epsilon(x)$ is arbitrary, by expanding the exponential we find

$$\left\langle \frac{\delta S[\phi]}{\delta\phi(x)} \right\rangle_J = J(x)\,, \tag{2.2.37}$$

which for $J \to 0$ is just the quantum average of the classical equations of motion.

But Eq. (2.2.37) is more useful than that. If we leave the dependence on the source J intact and exploit (2.2.31) again, it becomes

$$\frac{1}{Z[J]}\,\frac{\delta S}{\delta\phi}\left[\frac{i\delta}{\delta J}\right] Z[J] = J(x)\,. \tag{2.2.38}$$

This should be read in the sense that we replace the functional dependence of $\delta S/\delta\phi$ on ϕ by a dependence on $i\delta/\delta J$ and apply it to the partition function. In this way, the equation serves as a 'generating DSE', because upon taking further functional derivatives $\delta^n/\delta J^n$ and setting $J \to 0$ in the end, we successively generate relations between the correlation functions (2.2.29) of the theory — the tower of DSEs. Note that the path integral no longer appears in these equations explicitly; instead, we calculate the correlation functions *from each other*.

As an example, we take another derivative $i\delta/\delta J$ of (2.2.31) to obtain

$$\begin{aligned}\frac{i\delta}{\delta J(y)}\,\langle f(\phi)\rangle_J &= \frac{1}{Z[J]}\,f\left[\frac{i\delta}{\delta J}\right]\frac{i\delta}{\delta J(y)}\,Z[J] - \frac{1}{Z[J]^2}\,\frac{i\,\delta Z[J]}{\delta J(y)}\,f\left[\frac{i\delta}{\delta J}\right] Z[J] \\ &= \langle f(\phi)\,\phi(y)\rangle_J - \langle f(\phi)\rangle_J\,\langle\phi(y)\rangle_J\,.\end{aligned} \tag{2.2.39}$$

Applied to Eq. (2.2.37) and setting the sources to zero, this yields

$$\left\langle \frac{\delta S[\phi]}{\delta\phi(x)}\,\phi(y) \right\rangle = i\delta^4(x-y)\,. \tag{2.2.40}$$

From here one can derive the DSEs for the n-point functions with a tree-level counterpart in the action (which enter with $\propto \phi^n$ terms). For example, in a free scalar theory the propagator DSE becomes

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -(\Box + m^2)\,\phi(x) \qquad \Rightarrow \qquad -(\Box_x + m^2)\,\langle\phi(x)\,\phi(y)\rangle = i\delta^4(x-y)\,, \tag{2.2.41}$$

which returns the Feynman propagator $i/(p^2 - m^2 + i\epsilon)$ in momentum space.

Similar types of equations for the correlation functions can be derived from symmetry relations following from the Noether theorem. These are the **Ward-Takahashi identities (WTIs)** and **Slavnov-Taylor identities (STIs)**, which we will briefly touch upon in Sec. 2.2.3 and discuss in more detail in Sec. 3.1.2.

Quantum effective action. The relations above are rather formal and generic. What do they look like in practice? At this point it is useful to distinguish between

- the full correlation functions, generated by the partition function $Z[J]$,
- **connected** correlation functions, which enter in S-matrix elements and are thus of physical interest,
- and **one-particle-irreducible (1PI)** correlation functions, where external propagators are amputated and we keep only those diagrams that do not fall apart by cutting one line (see Fig. 2.9). In this way, they do away with the redundancy and describe the irreducible content of an $n-$point interaction vertex. For example, renormalizability can be determined from the 1PI vertices alone; hence they are the prime quantities of theoretical interest.

We also define two new generating functionals, which generate these new types of correlation functions by functional derivatives, namely $W[J] = -i \ln Z[J]$ and the **quantum effective action**

$$\Gamma[\varphi] = W[J] + \int d^4x \, \varphi(x) \, J(x) \, . \tag{2.2.42}$$

$W[J]$ depends on the source field $J(x)$, whereas the source that appears in $\Gamma[\varphi]$ is the **averaged field** $\varphi(x)$. From Eq. (2.2.42) one can see that $W[J]$ and $\Gamma[\varphi]$ are Legendre transforms of each other. The meaning of these quantities becomes more clear if we write the partition function as

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\phi \, e^{i\left(S[\phi] - \int d^4x \, \phi(x) J(x)\right)} = e^{i\left(\Gamma[\varphi] - \int d^4x \, \varphi(x) \, J(x)\right)} \, . \tag{2.2.43}$$

In this sense, $\Gamma[\varphi]$ and $\varphi(x)$ can be seen as the quantum versions of the classical action $S[\phi]$ and the classical field $\phi(x)$, integrated over quantum fluctuations and with the path integral exponential as a weight factor. More precisely, from Eq. (2.2.42) we have

$$\varphi(x) = -\frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z[J]} \frac{i\delta}{\delta J(x)} Z[J] \overset{(2.2.31)}{=} \langle \phi(x) \rangle_J \, . \tag{2.2.44}$$

Thus, $\varphi(x)$ is the vacuum expectation value of the field $\phi(x)$ in the presence of the source J, which vanishes[2] in the limit $J = 0$. Vice versa, $J(x)$ satisfies

$$J(x) = \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} \, . \tag{2.2.45}$$

Like in thermodynamics, $J(x)$ and $\varphi(x)$ are conjugate variables and the generating functionals $W[J]$ and $\Gamma[\varphi]$ are the corresponding potentials. J is the 'intensive' and φ the 'extensive' variable, and differentiation with respect to one gives the other.

While the classical action $S[\phi]$ contains the content of the classical field theory, either of the functionals $Z[J]$, $W[J]$ or $\Gamma[\varphi]$ determines the QFT completely since all

[2] Unless in the case of spontaneous symmetry breaking, but even then one can redefine the field so that its vacuum expectation value vanishes.

correlation functions can be derived from them: the connected ones are derivatives of $W[J]$ with respect to J, and 1PI vertices are derivatives of the effective action $\Gamma[\varphi]$ with respect to φ. For example, the 1PI two- and three-point functions are given by

$$\left.\frac{\delta^2\,\Gamma[\varphi]}{\delta\varphi(x_1)\,\delta\varphi(x_2)}\right|_{\varphi=0}, \qquad \left.\frac{\delta^3\,\Gamma[\varphi]}{\delta\varphi(x_1)\,\delta\varphi(x_2)\,\delta\varphi(x_3)}\right|_{\varphi=0}. \tag{2.2.46}$$

Relations between full, connected and 1PI correlation functions. To relate the full correlation functions with their connected and 1PI counterparts, we must reformulate Eq. (2.2.31) in terms of $W[J]$ instead of $Z[J]$. To do so, we exploit the useful relation

$$e^{-A} f(\partial)\, e^{A} = f(\partial + \partial A)\,, \tag{2.2.47}$$

where ∂ stands for a generic derivative acting on A. The r.h.s. acts on 1, so all unsaturated derivatives vanish. It is straightforward to verify this for polynomial functions, e.g. $f(\partial) = \partial^2$:

$$\begin{aligned} e^{-A}\,\partial^2\, e^{A} &= e^{-A}\,\partial\,(A'\,e^{A}) = A'' + A'^2\,, \\ (\partial + \partial A)^2 &= (\partial + \partial A)\,\partial A = A'' + A'^2\,. \end{aligned} \tag{2.2.48}$$

Applied to Eq. (2.2.31), with $A = iW[J]$ and $\partial = i\delta/\delta J$, we arrive at

$$\langle f[\phi]\rangle_J = f\left[\frac{i\delta}{\delta J} - \frac{\delta W[J]}{\delta J}\right]. \tag{2.2.49}$$

Here the dependence of f on each field value $\varphi(x_i)$ has to be replaced by a dependence on the bracket above, with $\delta/\delta J(x_i)$, and all unsaturated derivatives vanish since it acts on 1. In this way we have expressed the vacuum expectation value of $f(\phi)$ through functional derivatives of $W[J]$, which are the connected n-point functions. For example, if we abbreviate

$$W''_{xy}[J] = \frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)}\,, \qquad \Gamma''_{xy}[\varphi] = \frac{\delta^2 \Gamma[\varphi]}{\delta\varphi(x)\,\delta\varphi(y)}\,, \qquad \text{etc.} \tag{2.2.50}$$

we obtain for the two-point function:

$$\begin{aligned} \langle\phi(x)\,\phi(y)\rangle_J &= \left(\frac{i\delta}{\delta J(x)} - W'_x[J]\right)\left(\frac{i\delta}{\delta J(y)} - W'_y[J]\right) \\ &= W'_x[J]\,W'_y[J] - i\,W''_{xy}[J] \\ &= \varphi(x)\,\varphi(y) - i\,W''_{xy}[J]\,. \end{aligned} \tag{2.2.51}$$

The unsaturated derivative in the first line vanishes, and in the third line we used Eq. (2.2.44). In the limit $J = 0$ the vacuum expectation value $\varphi(x)$ is zero, and we find that the 'connected' propagator $W''_{xy}[0]$ is the same as the usual one, $\langle\phi(x)\,\phi(y)\rangle$, modulo a factor i. The same exercise can be repeated for higher n-point functions to express them in terms of connected and disconnected parts.

Now let us reformulate Eq. (2.2.49) in terms of $\Gamma[\varphi]$ instead of $W[J]$. From Eqs. (2.2.44–2.2.45) we have $W'_x[J] = -\varphi(x)$ and $\Gamma'_x[\varphi] = J(x)$ and therefore

$$\int d^4y\, W''_{xy}[J]\,\Gamma''_{yz}[\varphi] = -\int d^4y\,\frac{\delta\varphi(x)}{\delta J(y)}\,\frac{\delta J(y)}{\delta\varphi(z)} = -\frac{\delta\varphi(x)}{\delta\varphi(z)} = -\delta^4(x-z)\,. \tag{2.2.52}$$

In other words, the 1PI two-point function $\Gamma''_{xy}[0]$ is the inverse of $W''_{xy}[0]$. This explains why the *inverse* tree-level propagators appear in the classical action together with the tree-level vertices: the classical action contains the *1PI correlation functions* at tree level.

Let us denote the dressed propagator in the presence of the external source φ by $\Delta_{xy}[\varphi] = \Gamma''_{xy}[\varphi]^{-1}$. The product rule entails that the derivative of the propagator with respect to φ is

$$\frac{\delta}{\delta\varphi(z)}\,\Delta_{xy}[\varphi] = -\int d^4a \int d^4b\,\Delta_{xa}[\varphi]\,\Gamma'''_{azb}[\varphi]\,\Delta_{by}[\varphi]. \tag{2.2.53}$$

The derivative with respect to J becomes

$$\frac{\delta}{\delta J(x)} = \int d^4y\,\frac{\delta\varphi(y)}{\delta J(x)}\,\frac{\delta}{\delta\varphi(y)} = -\int d^4y\, W''_{xy}[J]\,\frac{\delta}{\delta\varphi(y)} = \int d^4y\,\Delta_{xy}[\varphi]\,\frac{\delta}{\delta\varphi(y)}\,, \tag{2.2.54}$$

so we can express Eq. (2.2.49) in terms of the effective action and its derivatives:

$$\langle f[\phi]\rangle_J = f\left[\varphi(x) + \int d^4y\,\Delta_{xy}[\varphi]\,\frac{i\delta}{\delta\varphi(y)}\right]. \tag{2.2.55}$$

This is the identity that we were after, because it allows us to relate the full n-point functions with their 1PI counterpart. Evaluating it for the two-point function, we recover our previous result (2.2.51):

$$\langle\phi(x)\,\phi(y)\rangle_J = \left(\varphi(x) + \int d^4a\,\Delta_{xa}[\varphi]\,\frac{i\delta}{\delta\varphi(a)}\right)\varphi(y) = \varphi(x)\,\varphi(y) + i\Delta_{xy}[\varphi]\,. \tag{2.2.56}$$

For the three-point function we obtain

$$\begin{aligned}\langle\phi(x)\,\phi(y)\,\phi(z)\rangle_J &= \left(\varphi(x) + \int d^4a\,\Delta_{xa}[\varphi]\,\frac{i\delta}{\delta\varphi(a)}\right)\left(\varphi(y) + \int d^4b\,\Delta_{yb}[\varphi]\,\frac{i\delta}{\delta\varphi(b)}\right)\varphi(z)\\ &= \left(\varphi(x) + \int d^4a\,\Delta_{xa}[\varphi]\,\frac{i\delta}{\delta\varphi(a)}\right)(\varphi(y)\,\varphi(z) + i\Delta_{yz}[\varphi])\\ &= \varphi(x)\,\varphi(y)\,\varphi(z) + i\Delta_{xy}[\varphi]\,\varphi(z) + i\Delta_{xz}[\varphi]\,\varphi(y) + i\Delta_{yz}[\varphi]\,\varphi(x)\\ &\quad + \int d^4a\int d^4b\int d^4c\,\Delta_{xa}[\varphi]\,\Delta_{yb}[\varphi]\,\Gamma'''_{bac}[\varphi]\,\Delta_{cz}[\varphi]\,,\end{aligned} \tag{2.2.57}$$

where we used Eq. (2.2.53) in the final step. After setting the sources to zero, only the last line survives and we find that the full three-point function is just the 1PI three-point vertex with external propagator legs attached. Going further to the four-point function, the relation would become more complicated since the four-point function has 1-particle-reducible topologies that fall apart by cutting one line (see below).

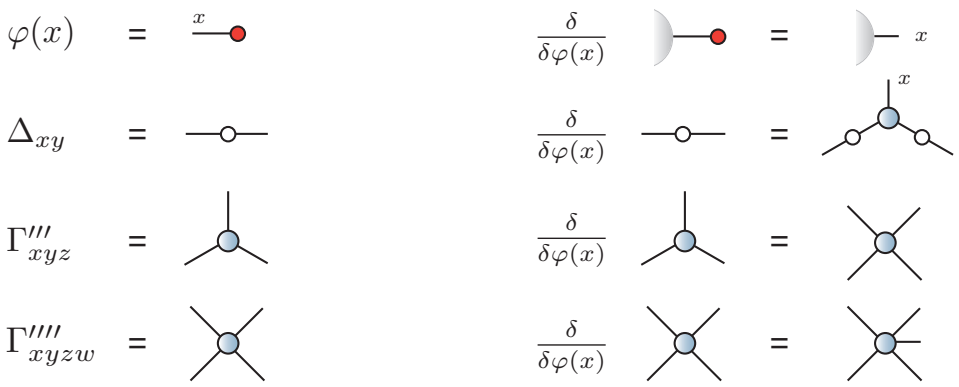
Dyson-Schwinger equations for 1PI correlation functions. Eq. (2.2.55) is not only useful for relating full and 1PI correlation functions, but also for working out the quantum versions of classical equations $f(\phi) = 0$ between the fields: we just need to replace the (usually non-linear) dependence on ϕ by the bracket on the r.h.s. – which generates further derivatives and derivatives of propagators – and set all fields to zero in the end, together with all unsaturated derivatives. If the classical action contains more than one field, then the functional dependence holds for each $\varphi_i(x)$, and the integral over y also goes over all intermediate (mixed!) propagators, which drop out in the end when setting the sources to zero.

With this we can express the generating DSE (2.2.37) in terms of the 1PI correlation functions:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = \frac{\delta S}{\delta\phi}\left[\varphi(x) + \int d^4y\; \Delta_{xy}[\varphi]\,\frac{i\delta}{\delta\varphi(y)}\right]. \tag{2.2.58}$$

Now we can better see what the equation actually means: On the r.h.s. we have the classical equations of motion, but with the ϕ dependence replaced by the content of the bracket. Like in Eqs. (2.2.56) and (2.2.57), this will reproduce the classical equations of motion for $\varphi(x)$ plus further terms which describe quantum corrections. The l.h.s. is the derivative of the quantum effective action. If we take further derivatives, we generate the 1PI two-, three-, ... n-point functions, which therefore always contain a classical part together with quantum contributions.

In practice Eq. (2.2.55) amounts to repeated applications of the bracket with the derivative, which suggests to use a **diagrammatic language**. We need graphical expressions for the source field $\varphi(x)$, the propagator Δ_{xy}, the higher $n-$point functions Γ'''_{xyz}, Γ''''_{xyzw}, ... and the effect of the functional derivative $\delta/\delta\varphi(x)$ on these quantities:

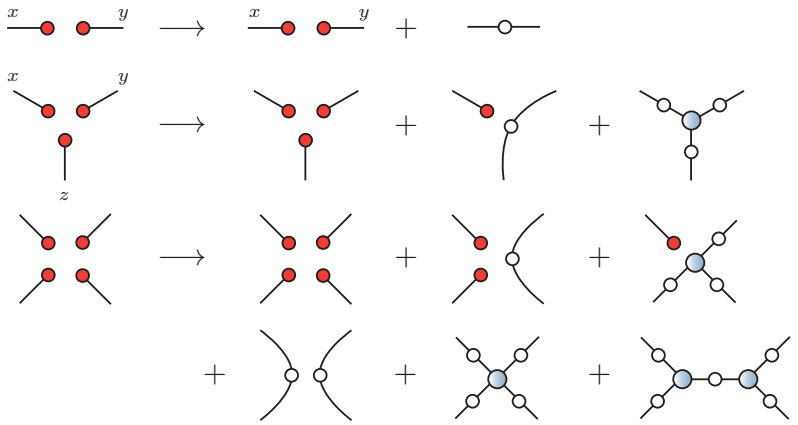


and so on. In the graphical notation we no longer distinguish between a correlation function that depends on $\varphi(x)$ as opposed to one where the field is set to zero, and for simplicity we also suppress all minus signs, i factors and multiplicities that arise from derivatives.

What we still need is a graphical analogue for Eq. (2.2.55), where $\phi(x)$ is replaced by $\varphi(x) + \int d^4y\, \Delta_{xy}\, i\delta/\delta\varphi(y)$. If we work this out for products of fields like in Eqs. (2.2.56) and (2.2.57) we arrive at:

$$\varphi(x)\ldots\varphi(z) \quad\to\quad \left(\varphi(x) + \int \ldots\right)\ldots\left(\varphi(z) + \int \ldots\right), \tag{2.2.59}$$

or graphically:



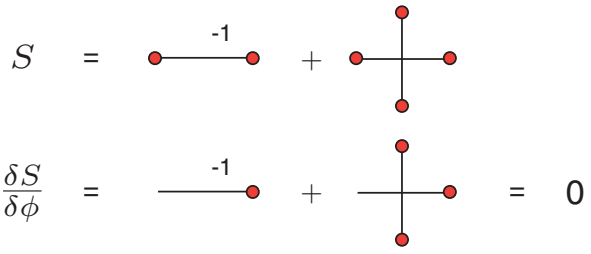
For simplicity we also absorbed the different symmetrizations with the same topology into one diagram. If we set $\varphi = 0$, these graphs tell us how ordinary n-point functions are related with their 1PI counterparts: the three-point functions are identical except for the external propagator legs, whereas the full four-point function is the sum of disconnected parts, a 1PI term, and 1-particle-reducible diagrams that contain 1PI three-point functions.

On the other hand, we can interpret these diagrams also differently: without the extra integral term in Eq. (2.2.55) we would return to the classical quantity $f(\varphi)$ expressed in terms of φ instead of ϕ. In the graphical notation we can then also drop the distinction between $\phi(x)$ and its quantum expectation value $\varphi(x)$ and use the same symbols for the fields that appear in the Lagrangian. In that way we can transform equations for the classical fields (equations of motion, symmetry relations etc.) into quantum identities for the 1PI correlation functions. Going from 'classical' to 'quantum' in the picture above then entails to connect the legs in all possible ways and equip them with dressed propagators.

ϕ^4 theory. We illustrate this by considering the simplest scalar field theory, ϕ^4 theory. The classical action and its functional derivative are given by

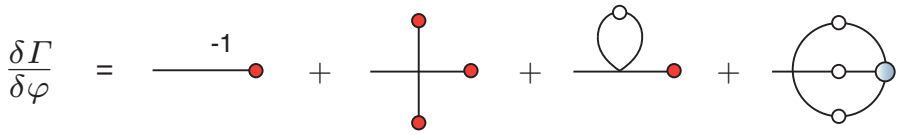
$$S = \int d^4x \left[\frac{1}{2}\left(\partial^\mu\phi\,\partial_\mu\phi - m^2\phi^2\right) - \frac{g}{4!}\,\phi^4\right], \qquad \frac{\delta S}{\delta\phi} = -(\Box + m^2)\phi - \frac{g}{3!}\,\phi^3\,. \quad (2.2.60)$$

Setting $\delta S/\delta\phi = 0$ yields the classical equations of motion. Diagrammatically, this amounts to



The line with '-1' is the inverse tree-level propagator. In the classical action it is contracted with the field ϕ; the functional derivative removes one instance of ϕ. As before we ignore all prefactors and multiplicities.

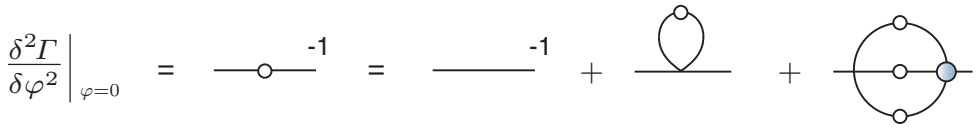
Now 'connect the dots' in all possible ways to obtain the quantum eq. of motion:



We arrive at the same result in formulas if we replace ϕ in Eq. (2.2.60) with the bracket in (2.2.58) and let it act on 1:

$$\frac{\delta\Gamma}{\delta\varphi(x)} = -(\Box + m^2)\,\varphi(x) - \frac{g}{3!}\left[\varphi(x)^3 + 3i\,\varphi(x)\,\Delta_{xx} + \iiint\limits_{z\,z'z''} \Delta_{xz}\Delta_{xz'}\Delta_{xz''}\Gamma'''_{zz'z''}\right], \tag{2.2.61}$$

where we suppressed the arguments in $\Delta[\widetilde{\varphi}]$ and $\Gamma'''[\widetilde{\varphi}]$ and abbreviated the integrals $\int d^4z$ by $\int_z$. Applying another derivative and setting $\varphi = 0$ yields the DSE for the inverse scalar propagator:



$$\Delta^{-1}_{xy} = -(\Box + m^2)\,\delta^4(x-y) - \frac{g}{3!}\left[3i\,\delta^4(x-y)\,\Delta_{xx} + \iiint\limits_{z\,z'z''} \Delta_{xz}\Delta_{xz'}\Delta_{xz''}\Gamma''''_{zz'z''y}\right]. \tag{2.2.62}$$

In principle, the derivatives of the propagators in the two-loop diagram produce further terms including three-point vertices; however, as a consequence of the Lagrangian's Z_2 symmetry under $\varphi \to -\varphi$, there are no odd n-point functions in ϕ^4 theory and therefore $\Gamma'''_{xyz}[0] = 0$.

Eq. (2.2.62) says that the inverse full ('dressed') propagator is the sum of the inverse classical (tree-level) propagator plus quantum loop corrections, which define the self-energy. The equation is exact but depends on the four-point vertex, which satisfies its own DSE. Still, if we happened to *know* the exact four-point vertex, the equation would give us the exact two-point function. This goes back to the comment below Eq. (2.2.38): instead of working out the path integral explicitly, one can calculate the correlation functions from each other. In practice the equations are solved in momentum space, where each loop becomes a four-momentum integration, so they have the usual structure of Feynman diagrams.

Since the DSEs are exact, they also reproduce **perturbation theory** if the coupling is small. To get a DSE for the propagator instead of its inverse, multiply Eq. (2.2.62) with the tree-level propagator (Δ_0) from the left and the full propagator (Δ) from the right (or vice versa):

$$\Delta^{-1} = \Delta_0^{-1} - \Sigma \qquad \Rightarrow \qquad \Delta = \Delta_0 + \Delta_0\,\Sigma\,\Delta\,, \tag{2.2.63}$$

where Σ is the sum of the self-energy terms. Reinsert the equation again for each instance of the dressed propagator on the right,

$$\Delta = \Delta_0 + \Delta_0\,\Sigma\,\Delta = \Delta_0 + \Delta_0\,\Sigma\,\Delta_0 + \Delta_0\,\Sigma\,\Delta\,\Sigma\,\Delta = \dots\,, \tag{2.2.64}$$

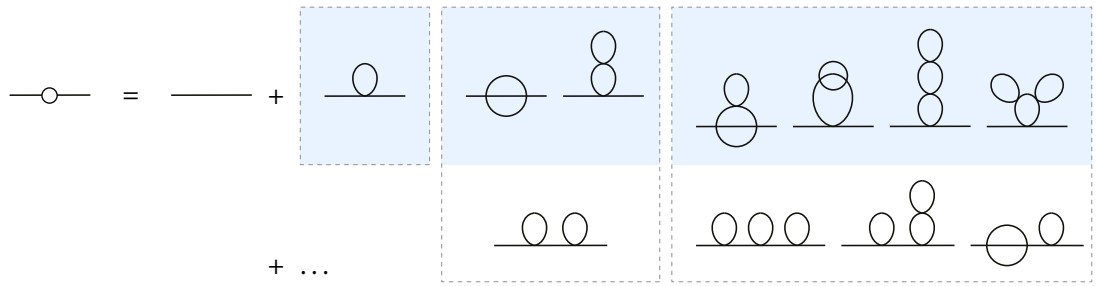


FIG. 2.10: Perturbative expansion of the propagator in ϕ^4 theory. The colored boxes highlight the 1PI diagrams which appear in the self-energy.

and do the same for every instance of the propagator and the vertex inside the self-energy Σ. Because the self-energy comes with a factor g and the four-point vertex includes another factor g, every additional loop carries a factor g; and as long as $|g| < 1$, higher loop diagrams are suppressed. In this way we successively generate the perturbative series for the propagator (Fig. 2.10).

Even if the perturbation series does not converge, it is useful to remember that there are non-perturbative, exact DSEs behind it, which can be derived for any QFT. A simple analogue is the geometric series: The solution of the equation

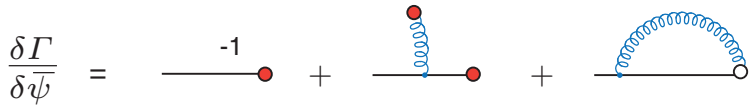
$$f(x) = 1 + xf(x) = 1 + x + x^2 f(x) = 1 + x + x^2 + x^3 f(x) = \dots , \tag{2.2.65}$$

is $f(x) = 1/(1-x)$. The geometric series $f(x) = \sum_{n=0}^{\infty} x^n$ converges to that result only if $|x| < 1$, whereas each step in Eq. (2.2.65) is non-perturbative and exact. The difference comes from the last term, where $f(x)$ appears again and pulls the result back even if x becomes explosively large.

Dyson-Schwinger equations in QCD. Let us be bold and try to apply the procedure to QCD right away. There are still issues we have not yet dealt with (see Sec. 2.2.3), which will produce additional ghost diagrams, but otherwise the diagrammatic derivation goes through as before. The classical action of QCD takes the form:

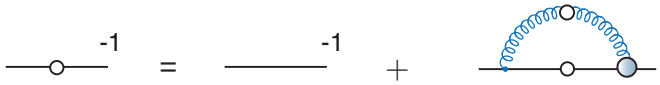
We use the convention that the left circles in the fermion terms represent the antiquark fields $\bar{\psi}$ and the right circles the quark fields ψ. Taking a functional derivative with respect to $\bar{\psi}$ yields the classical equation of motion for the quark, the Dirac equation:

Consequently, the quantum equation of motion becomes (connect the dots)



Note that this involves a mixed, field-dependent 'quark-gluon propagator' on the right. Taking another functional derivative with respect to ψ and setting all fields to zero gives us the **quark DSE** for the inverse quark propagator:

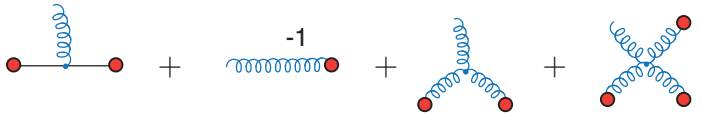
$$\left.\frac{\delta^2\Gamma}{\delta\bar{\psi}\delta\psi}\right|_{A,\psi,\bar{\psi}=0} =$$



The quark DSE tells us that the inverse full propagator is the sum of the inverse tree-level propagator plus quantum loop corrections, which are contained in the quark self-energy. In practice it enables us to compute the quark propagator if the gluon propagator and quark-gluon vertex are known.

We can repeat the same steps also for the gluon propagator. The classical equation of motion for the gluon (the Maxwell equation) reads

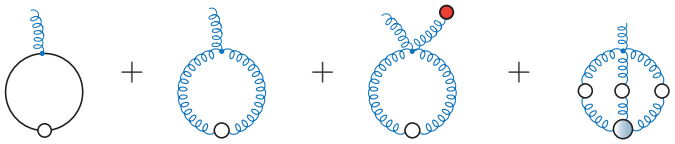
$$\frac{\delta S}{\delta A} =$$



$$= 0$$

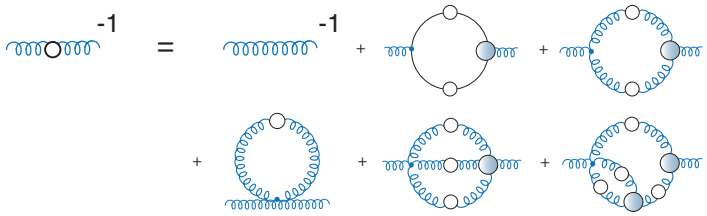
and the corresponding quantum equation of motion:

$$\frac{\delta\Gamma}{\delta A} = \frac{\delta S}{\delta A} +$$



After taking another derivative with respect to A and setting all sources to zero, we obtain the **gluon DSE:**

$$\left.\frac{\delta^2\Gamma}{\delta A^2}\right|_{A,\psi,\bar{\psi}=0} =$$



The inverse dressed gluon propagator is the sum of the inverse tree-level propagator, a quark loop, a gluon loop, a tadpole diagram, a 'sunset' diagram and a 'squint' diagram. The gauge fixing procedure in the next section will also produce a ghost loop, which has the same form as the quark loop. The inputs of the equation are then the quark and ghost propagators and the quark-gluon, ghost-gluon, three-gluon and four-gluon vertices. If we solved the quark, gluon and ghost DSEs in combination (i.e., all two-point functions), the remaining inputs would be the vertices.

Actually we have ignored another subtlety: if the action contains several fields, one has to sum over them when taking the derivative of the propagators in Eq. (2.2.53), which also leads to mixed vertices. The general rule is that, after taking functional derivatives, for each internal 'half-propagator' connected to a dressed vertex one has to sum over all types of fields. This does not modify the quark and gluon DSEs but it will produce additional diagrams, for example, in the quark-gluon vertex DSE.

2.2.3 Gauge fixing in QCD

Path integrals in QCD. We have discussed most of the things so far at the level of a generic QFT, so we should get a bit more specific. The straightforward generalization of the partition function in Eq. (2.2.25) to QCD with quark (ψ), antiquark ($\bar{\psi}$) and gluon fields (A^μ) is:

$$Z[J,\bar{\eta},\eta] = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\; e^{i\left(S[A,\psi,\bar{\psi}]-\int d^4x\,(J_\mu A^\mu+\bar{\eta}\,\psi+\eta\,\bar{\psi})\right)}\,, \tag{2.2.66}$$

where we have added a vector source J^μ for the gluon field and spinor sources $\bar{\eta}$ and η for the quark and antiquark fields, respectively. The fermion fields in the path integral are anticommuting Grassmann numbers, whereas their corresponding field operators in the canonical approach satisfy equal-time anticommutation relations:

$$\begin{aligned}\left\{\psi_{\alpha,i}(x),\psi^\dagger_{\beta,j}(y)\right\}_{x^0=y^0} &= \delta^3(\boldsymbol{x}-\boldsymbol{y})\,\delta_{\alpha\beta}\,\delta_{ij},\\ \left\{\psi_{\alpha,i}(x),\psi_{\beta,j}(y)\right\}_{x^0=y^0} &= \left\{\psi^\dagger_{\alpha,i}(x),\psi^\dagger_{\beta,j}(y)\right\}_{x^0=y^0} = 0\,.\end{aligned} \tag{2.2.67}$$

For example, the **quark propagator** is given by

$$S_{\alpha\beta}(x_1,x_2) = \langle 0|\mathsf{T}\,\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)|0\rangle = \frac{i^2\delta^2}{\delta\bar{\eta}_\alpha(x_1)\,\delta\eta_\beta(x_2)}\bigg|_{J,\,\eta,\,\bar{\eta}=0}\frac{Z[J,\eta,\bar{\eta}\,]}{Z[\,0,0,0\,]}\,. \tag{2.2.68}$$

We recollect some relations for **Grassmann variables**. A Grassmann algebra is an algebra generated by a basis $\{\theta_1\,\ldots\theta_n\}$ which satisfies anticommutation relations $\{\theta_i,\theta_j\}=0$. In particular, this implies $\theta_i^2=0$. As a consequence, a general element of the algebra is at most linear in each θ_i,

$$f(\theta) = c_0 + c_1\,\theta\,, \qquad f(\theta_1,\theta_2) = c_0 + c_1\,\theta_1 + c_2\,\theta_2 + c_{12}\,\theta_1\,\theta_2\,, \qquad \text{etc.} \tag{2.2.69}$$

which also means that a Taylor expansion stops: $e^{a\theta} = 1 + a\theta$. A derivative can then be defined by replacing $\theta_i \to 1$, but the θ_i must first be permuted to the derivative operator:

$$\frac{\partial f(\theta)}{\partial\theta} = c_1\,, \qquad \frac{\partial f(\theta_1,\theta_2)}{\partial\theta_1} = c_1 + c_{12}\,\theta_2\,, \qquad \frac{\partial f(\theta_1,\theta_2)}{\partial\theta_2} = c_2 - c_{12}\,\theta_1\,, \qquad \text{etc.} \tag{2.2.70}$$

The integration can be defined by

$$\int d\theta\,1 = 0\,, \qquad \int d\theta\,\theta = 1\,, \tag{2.2.71}$$

where the first relation guarantees translation invariance, $\int d\theta\,f(\theta+\eta) - \int d\theta\,f(\theta) = \int d\theta\,(c_1\,\eta) \stackrel{!}{=} 0$, and the second is a normalization. Also here the integration variable must be permuted to the integral measure. As a consequence, the integration and derivative are the same:

$$\int d\theta\,f(\theta) = \int d\theta\,(c_0 + c_1\,\theta) = c_1 = \frac{df(\theta)}{d\theta}\,. \tag{2.2.72}$$

An integral over n Grassmann variables $d^n\theta = d\theta_n\ldots d\theta_1$ becomes

$$\int d^n\theta\,\theta_1\ldots\theta_n = 1\,, \qquad \int d^n\theta\,\theta_{i_1}\ldots\theta_{i_n} = \varepsilon_{i_1\ldots i_n}\,, \tag{2.2.73}$$

where $\varepsilon_{i_1\ldots i_n}$ is the totally antisymmetric tensor with $\varepsilon_{1\ldots n}=1$. In the integral over $f(\boldsymbol{\theta}) = f(\theta_1,\ldots\theta_n)$ only the last term in Eq. (2.2.69) survives, since every $d\theta_i$ must be saturated by θ_i:

$$\int d^n\theta\,f(\boldsymbol{\theta}) = \int d^n\theta\,c_{1\ldots n}\,\theta_1\ldots\theta_n = c_{1\ldots n} \tag{2.2.74}$$

To obtain the Jacobian for the transformation $\theta'_i = A_{ij}\,\theta_j$, we write

$$\int d^n\theta\, f(\boldsymbol{\theta}') = \int d^n\theta\, c_{1\ldots n}\, A_{1i_1}\ldots A_{ni_n}\,\theta_{i_1}\ldots\theta_{i_n} = \underbrace{\varepsilon_{1\ldots n}\, A_{1i_1}\ldots A_{ni_n}}_{=\det A}\, c_{1\ldots n}\,, \tag{2.2.75}$$

and since $c_{1\ldots n} = \int d^n\theta\, f(\boldsymbol{\theta}) = \int d^n\theta'\, f(\boldsymbol{\theta}')$ we find

$$d^n\theta' = \frac{1}{\det A}\, d^n\theta\,. \tag{2.2.76}$$

This is in contrast to the bosonic case, where $x'_i = A_{ij}\,x_j$ leads to $d^nx' = |\det A|\, d^nx$.

We define complex Grassmann variables by $(\theta_i\,\theta_j)^* = \theta_j^*\,\theta_i^*$, so that

$$d^n\theta\, d^n\theta^* = d\theta_n\ldots d\theta_1\, d\theta_1^*\ldots d\theta_n^* = d\theta_1\, d\theta_1^*\ldots d\theta_n\, d\theta_n^* = d\theta_n\, d\theta_n^*\ldots d\theta_1\, d\theta_1^*\,. \tag{2.2.77}$$

Pairs like $d\theta_i\, d\theta_i^*$ are bosonic and can be permuted through, whereas $d^n\theta\, d^n\theta^* = (-1)^n\, d^n\theta^*\, d^n\theta$. This yields $\int d^n\theta\, d^n\theta^*\,\theta_{i1}^*\,\theta_{j1}\ldots\theta_{i_n}^*\,\theta_{j_n} = \varepsilon_{i_1\ldots i_n}\,\varepsilon_{j_1\ldots j_n}$, and in the integral $\int d^n\theta\, d^n\theta^*\, f(\boldsymbol{\theta},\boldsymbol{\theta}^*)$ only the term $\propto \theta_1\ldots\theta_n\,\theta_1^*\ldots\theta_n^*$ survives. Finally, we calculate the integral over a Gaussian:

$$\begin{aligned}\int d^n\theta\, d^n\theta^*\, e^{\boldsymbol{\theta}^*B\,\boldsymbol{\theta}} &= \int d^n\theta\, d^n\theta^*\,\frac{1}{n!}\,(\boldsymbol{\theta}^*B\,\boldsymbol{\theta})^n \\ &= \frac{1}{n!}\,B_{i_1j_1}\ldots B_{i_nj_n}\int d^n\theta\, d^n\theta^*\,\theta_{i1}^*\,\theta_{j1}\ldots\theta_{i_n}^*\,\theta_{j_n} \\ &= \frac{1}{n!}\,\varepsilon_{i_1\ldots i_n}\,\varepsilon_{j_1\ldots j_n}\,B_{i_1j_1}\ldots B_{i_nj_n} = \det B\,.\end{aligned} \tag{2.2.78}$$

If the exponent comes with a minus sign, this will produce a factor $(-1)^n$ which can be compensated by employing the integral measure $d^n\theta^*\, d^n\theta$:

$$\int d^n\theta^*\, d^n\theta\, e^{-\boldsymbol{\theta}^*B\,\boldsymbol{\theta}} = \det B\,. \tag{2.2.79}$$

The fact that we can express the determinant of a matrix as an integral over Grassmann variables will become extremely useful in a moment. Another useful relation is

$$\begin{aligned}\int d^n\theta^*\, d^n\theta\, e^{-(\boldsymbol{\theta}^*B\,\boldsymbol{\theta}+\boldsymbol{\eta}^*\boldsymbol{\theta}+\boldsymbol{\eta}\,\boldsymbol{\theta}^*)} &= e^{\boldsymbol{\eta}^*B^{-1}\boldsymbol{\eta}}\int d^n\theta^*\, d^n\theta\, e^{-(\boldsymbol{\theta}^*+\boldsymbol{\eta}^*B^{-1})\,B\,(\boldsymbol{\theta}+B^{-1}\boldsymbol{\eta})} \\ &= e^{\boldsymbol{\eta}^*B^{-1}\boldsymbol{\eta}}\,\det B\,,\end{aligned} \tag{2.2.80}$$

from where one can verify Eq. (2.2.68) for the tree-level quark propagator if one starts from the partition function of a free fermion Lagrangian.

The **gluon propagator** is given by

$$D^{\mu\nu}(x_1,x_2) = \langle 0|\mathsf{T}\,A^\mu(x_1)A^\nu(x_2)|0\rangle = \frac{i^2\delta^2}{\delta J_\mu(x_1)\,\delta J_\nu(x_2)}\bigg|_{J,\,\eta,\,\bar\eta=0}\frac{Z[J,\eta,\bar\eta\,]}{Z[\,0,0,0\,]}\,, \tag{2.2.81}$$

but from the discussion around Eq. (2.1.21) this expression does not yet make sense. The problem appears in the kinetic gluon term in the Lagrangian:

$$-\tfrac{1}{4}F^a_{\mu\nu}\,F_a^{\mu\nu} \;\cong\; \frac{1}{2}\,A^a_\mu\,(\Box\,g^{\mu\nu}-\partial^\mu\partial^\nu)\,A^a_\nu + \ldots \tag{2.2.82}$$

The tree-level inverse gluon propagator in momentum space is proportional to a transverse projector,

$$D_0^{-1}(p)^{\mu\nu} = ip^2\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right), \tag{2.2.83}$$

but projectors are not invertible: if P_+ and P_- are generic projectors that satisfy $P_\pm^2 = P_\pm$, $P_+P_- = 0$ and $P_+ + P_- = 1$, then a linear combination of them has the form

$$F = \alpha\, P_+ + \beta\, P_- \qquad \Rightarrow \qquad F^{-1} = \frac{P_+}{\alpha} + \frac{P_-}{\beta}\,. \tag{2.2.84}$$

For $\alpha = 0$ or $\beta = 0$, the inverse is not well-defined. Applied to Eq. (2.2.83), this means that the gluon propagator $D_0^{\mu\nu}(p)$ does not exist.

The problem appears in the gauge-boson sector and is structurally the same in QCD and QED. In fact, it already arises in the free theory by observing that A_0^μ has no canonically conjugate momentum. The conjugate momentum is defined by

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial\,(\partial_0\, A^\mu)} = F_{\mu 0}\,, \tag{2.2.85}$$

which entails $\Pi_0 = 0$, but this contradicts the equal-time commutation relations

$$[A_\mu(x), \Pi_\nu(y)]_{x^0=y^0} = i g_{\mu\nu}\, \delta^3(\boldsymbol{x} - \boldsymbol{y})\,. \tag{2.2.86}$$

Redundancy in the path integral. Another manifestation of the problem is that in the path integral over the gauge fields,

$$Z = \int \mathcal{D}A\, e^{iS[A]}\,, \tag{2.2.87}$$

we integrate over redundant degrees of freedom that are connected by gauge transformations. This is easy to see in QED: If we split the gauge fields into transverse and longitudinal parts $A^\mu = A_T^\mu + A_L^\mu$, with $\partial_\mu A_T^\mu = 0$ and $A_L^\mu = \partial^\mu\, \Theta$, we find that

$$\begin{aligned} &(\Box\, g^{\mu\nu} - \partial^\mu\partial^\nu)\, A_\nu^L = \Box\, \partial^\mu \Theta - \partial^\mu\, \Box \Theta = 0 \\ \Rightarrow\quad &(\Box\, g^{\mu\nu} - \partial^\mu\partial^\nu)\, A_\nu = (\Box\, g^{\mu\nu} - \partial^\mu\partial^\nu)\, A_\nu^T = \Box\, A_T^\mu \end{aligned} \tag{2.2.88}$$

and thus A_L^μ drops out completely from the kinetic term in the QED Lagrangian:

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} A_\mu\, (\Box g^{\mu\nu} - \partial^\mu\partial^\nu)\, A_\nu = \frac{1}{2} A_\mu^T\, \Box\, A_T^\mu\,. \tag{2.2.89}$$

Now, if we perform a gauge transformation

$$A'^\mu = U A^\mu U^\dagger + \frac{i}{g}\, U(\partial^\mu U^\dagger) = A^\mu + \frac{1}{g}\, \partial^\mu \varepsilon = A_T^\mu + \partial^\mu \left(\Theta + \frac{\varepsilon}{g}\right) \tag{2.2.90}$$

and compare to $A'^\mu = A'^\mu_T + \partial^\mu \Theta'$, we see that the gauge transformation only affects A_L^μ:

$$A'^\mu_T = A_T^\mu\,, \qquad \Theta' = \Theta + \frac{\varepsilon}{g}\,. \tag{2.2.91}$$

In other words, the path integral $\mathcal{D}A$ overcounts physically equivalent degrees of freedom, namely the longitudinal field components that emerge from each other by gauge transformations.

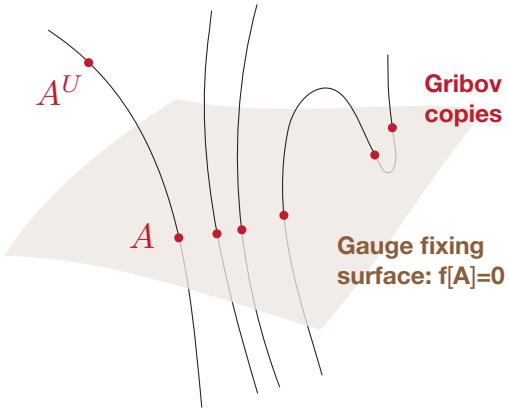


FIG. 2.11: Gauge orbits and gauge-fixing surface.

In QED the problem is cured by the **Gupta-Bleuler method**: The first step is to add a gauge-fixing term as a Lagrange multiplier with a gauge parameter ξ,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \overset{\text{p.I.}}{\cong} \frac{1}{2}A_\mu\left(\Box\, g^{\mu\nu} - \partial^\mu\partial^\nu + \frac{1}{\xi}\,\partial^\mu\partial^\nu\right)A_\nu \tag{2.2.92}$$

so that the inverse propagator picks up a longitudinal part and becomes invertible:

$$D_F^{-1}(p)^{\mu\nu} = ip^2\left[\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \frac{1}{\xi}\,\frac{p^\mu p^\nu}{p^2}\right]. \tag{2.2.93}$$

The second step is to impose the transversality condition $\langle\lambda|\partial_\mu A^\mu|\lambda\rangle$ for physical states $|\lambda\rangle$, which has the effect that the longitudinal and timelike photon polarizations cancel each other in S-matrix elements and external photon legs are always transverse. In fact, the need to preserve this feature when switching on interactions is the reason why we impose *local* gauge invariance also for fermions.

Faddeev-Popov gauge fixing. Unfortunately, the situation in QCD is more complicated because gauge transformations mix the transverse and longitudinal field components, and both of them contribute to the gluonic Lagrangian due to the three-gluon and four-gluon interactions.

The analogue of the Gupta-Bleuler method in the non-Abelian case is the **Faddeev-Popov method**. Let us denote a gauge transformation of the gluon field by $A \to A^U$, where U is some gauge transformation with gauge parameter ε. The basic idea is then to restrict the path integral

$$Z = \int \mathcal{D}A\, e^{iS[A]} \tag{2.2.94}$$

to the gauge-fixing surface

$$f[A] = \partial_\mu A^\mu = 0\,, \tag{2.2.95}$$

which is illustrated in Fig. 2.11. The gauge-fixing function $f[A]$ singles out a hypersurface of fixed gauge, so that each gauge field counts only once. In principle the condition $f[A]$ is arbitrary; the choice (2.2.95) corresponds to linear covariant gauges.

How would we implement such a condition? Presumably by inserting a delta function $\delta(f[A])$; however, this would modify the path integral. A better strategy is to insert a quantity that equals 1 but *contains* $\delta(f[A])$. Consider a one-dimensional example: suppose we have an integral over a variable ε (the 'gauge transformation parameter') and we want to restrict it to $f(\varepsilon) = 0$. Then the identity

$$\int d\varepsilon \left|\frac{df(\varepsilon)}{d\varepsilon}\right| \delta(f(\varepsilon)) = \int d\varepsilon |f'(\varepsilon_0)| \, \frac{\delta(\varepsilon - \varepsilon_0)}{|f'(\varepsilon_0)|} = 1 \tag{2.2.96}$$

holds as long as $f(\varepsilon) = 0$ has only one solution ε_0.

Let us generalize this relation to infinitely many variables, where ε becomes a continuous function $\varepsilon(x)$. In this case, the gauge transformation reads

$$(A^U)^\mu = U A^\mu U^\dagger + \frac{1}{g} U(\partial^\mu U^\dagger)\,, \qquad U(x) = e^{i \sum_a \varepsilon_a(x)\, t_a} \tag{2.2.97}$$

and the gauge-fixing condition is

$$f_a[A^U] = \partial_\mu (A^U_a)^\mu = \partial_\mu \left(A^\mu_a + \frac{1}{g} D^\mu_{ab}\, \varepsilon_b \right), \tag{2.2.98}$$

where $D^\mu_{ab} = \partial^\mu \delta_{ab} - g f_{abc} A^\mu_c$ is the covariant derivative in the adjoint representation from Eq. (2.1.34). Then the analogue of Eq. (2.2.96) is the 'functional unity'

$$\int \mathcal{D}U \, \det M[A] \, \delta(f[A^U]) = 1\,. \tag{2.2.99}$$

Here, the path integral

$$\mathcal{D}U = \lim_{n\to\infty} \prod_{i=1}^{n} \prod_a \varepsilon_a(x_i) \tag{2.2.100}$$

is the group volume, the $\delta-$function is an infinite product of $\delta-$functions at each space-time point x, and the **Faddeev-Popov operator** $M[A]$ is the functional derivative of the gauge-fixing condition with respect to the gauge transformation parameter:

$$\begin{aligned} M_{ab}[A](x,y) &= \left.\frac{\delta f_a[A^U](x)}{\delta \varepsilon_b(y)}\right|_{f[A^U]=0} \\ &= \frac{1}{g}\, \partial_\mu D^\mu_{ab}\, \delta^4(x-y) \\ &= \frac{1}{g}\, \delta_{ab}\, \Box\, \delta^4(x-y) - f_{abc}\, \partial_\mu \left(A^\mu_c\, \delta^4(x-y) \right). \end{aligned} \tag{2.2.101}$$

Note that $M[A]$ is independent of the gauge transformation parameter. In QED, it is also independent of A^μ because the second term with f_{abc} disappears, and thus $\det M[A]$ factorizes from the path integral and can be pulled out.

Now we can insert (2.2.99) in the path integral,

$$Z = \int \mathcal{D}A \int \mathcal{D}U \, \det M[A\,] \, \delta(f[A^U]) \, e^{iS[A]}\,, \tag{2.2.102}$$

and, because Z is gauge-invariant, perform a gauge transformation $A^U \to A$. The gauge field measure $\mathcal{D}A$, the group measure $\mathcal{D}U$, the Faddeev-Popov determinant and the classical action $S[A]$ are all invariant under this operation, so that it merely amounts to replacing $\delta(f[A^U]) \to \delta(f[A])$. The integrand then no longer depends on U and the group integration $\mathcal{D}U$ factorizes; it produces an infinite constant which drops out whenever we normalize Z, for example when calculating correlation functions. The remaining $\delta-$function restricts the integration over all fields to the hypersurface $f[A] = 0$. Each gauge orbit contributes only one field configuration and we have an integration over physically distinct fields.

The caveat here is that we have assumed the gauge-fixing condition to be unique, like in the one-dimensional example where we assumed that the equation $f(\varepsilon) = 0$ admits only one solution ε_0. This is usually not the case due to **Gribov copies**: the gauge-fixing condition can intersect the gauge orbits more than once and is therefore not complete (cf. Fig. 2.11); there is a residual gauge freedom. Also in QED $\partial_\mu A^\mu = \Box\,\Theta = 0$ does not fix Θ completely because Eq. (2.2.91) still allows us to perform residual gauge transformations as long as $\Box\,\varepsilon = 0$, but this freedom can be removed by imposing appropriate boundary conditions on the fields. In QCD it has been suggested to restrict the gauge fields to the interior of the Gribov horizon where $\det M[A] > 0$, or possibly even further, and there are indications that the properties close to the Gribov horizon might be related to confinement.

The remaining question is what to do with the Faddeev-Popov determinant and the $\delta-$function. It turns out that one can shuffle both quantities into the exponential and thus into the action $S[A]$. We can take care of the $\delta-$function by changing the gauge fixing condition to $f[A] + \frac{\xi}{2}\,B = 0$, where $B(x)$ lives in the Lie algebra but does not depend on A. This does not affect the Faddeev-Popov determinant, but the functional integral Z_B now implicitly depends on B. Since any B leads to the same gauge-invariant physics, we can work with Z_B, $Z_{B'}$ or $\int \mathcal{D}B\,F(B)\,Z_B$; these are all equivalent. If we integrate over the functions $B(x)$ with some Gaussian weight, we can remove the $\delta-$function in favor of a new term in the action:

$$\begin{aligned} Z &= \int \mathcal{D}B\, e^{-\frac{i\xi}{8}\int d^4x\, B^2(x)}\, \mathcal{D}A\, \det M[A]\, \delta\big(f[A] + \tfrac{\xi}{2}\,B\big)\, e^{iS[A]} \\ &= \int \mathcal{D}A\, \det M[A]\, e^{\,i\left(S[A] - \int d^4x\, \frac{f[A]^2}{2\xi}\right)}. \end{aligned} \tag{2.2.103}$$

With a linear covariant gauge, this provides just the same modification as the Gupta-Bleuler method in QED, Eq. (2.2.92):

$$-\tfrac{1}{4}F^a_{\mu\nu}\,F_a^{\mu\nu} - \frac{(\partial_\mu\,A_a^\mu)^2}{2\xi} \;\cong\; \frac{1}{2}\,A^a_\mu\left(\Box\,g^{\mu\nu} - \partial^\mu\partial^\nu + \frac{1}{\xi}\,\partial^\mu\partial^\nu\right)A^a_\nu + \ldots \tag{2.2.104}$$

As a result, the inverse gluon propagator is no longer transverse and can be inverted. ξ is the gauge parameter: $\xi = 0$ defines the Landau gauge, $\xi = 1$ the Feynman gauge, and in principle there are many other possible choices which differ not only by the gauge parameter but also by the gauge fixing condition (Coulomb gauge, axial gauge, light-cone gauge, maximal Abelian gauge etc.).

Finally, we want to shift the Faddeev-Popov determinant into the action as well. The trick is that the determinant of an operator can be written as a path integral over anticommuting Grassmann fields, cf. Eq. (2.2.78):

$$\det M[A] = \int \mathcal{D}c\,\mathcal{D}\bar{c}\; e^{\int d^4x \int d^4y\, \bar{c}_a(x)\, M_{ab}[A](x,y)\, c_b(y)}\,, \tag{2.2.105}$$

where the **Faddeev-Popov ghosts** $c_a(x)$, $\bar{c}_a(x)$ are scalar but Grassmann-valued fields. They carry the wrong Bose-Fermi statistics, but this is of no concern since they are anyway unphysical — they are just a consequence of fixing the gauge. Thus, the procedure is a generalization of the Gupta-Bleuler method since in QED the Faddeev-Popov determinant is independent of A^μ and can be pulled out of the path integral, so that only the $\delta-$function remains, whereas the non-Abelian case also requires the dynamical inclusion of ghost fields.

The new gauge-fixing terms in the action read

$$\begin{aligned} S_{\mathrm{GF}}[A,c,\bar{c}\,] &= -\int d^4x\,\frac{f[A]^2}{2\xi} - i\int d^4x \int d^4y\;\bar{c}_a(x)\,M_{ab}[A](x,y)\,c_b(y) \\ &\overset{\text{p.I.}}{\cong} \int d^4x\left[\frac{1}{2\xi}A^a_\mu\,\partial^\mu\partial^\nu A^a_\nu - \frac{i}{g}\,\bar{c}_a\,\Box\,c_a - i f_{abc}\,(\partial_\mu\,\bar{c}_a)\,A^\mu_c\,c_b\right] \\ &\cong \int d^4x\left[\frac{1}{2\xi}A^a_\mu\,\partial^\mu\partial^\nu A^a_\nu + \bar{c}_a\,\Box\,c_a - g f_{abc}\,(\partial_\mu\,\bar{c}_a)\,A^\mu_b\,c_c\right], \end{aligned} \tag{2.2.106}$$

where in the third line we absorbed the factor $-i/g$ into the antighost field. In conclusion, we obtained a longitudinal term in the gluon propagator, a massless ghost propagator, and a ghost-gluon three-point vertex with coupling constant g.

Reinstating the quarks and including all source terms, the final partition function of QCD takes the form

$$Z[J,\bar{\eta},\eta,\bar{\sigma},\sigma\,] = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}c\,\mathcal{D}\bar{c}\;e^{i\left(S[A,\psi,\bar{\psi}\,]+S_{\mathrm{GF}}[A,c,\bar{c}\,]+S_{\mathrm{C}}\right)}. \tag{2.2.107}$$

The source term contains the gluon source J^μ, quark sources η, $\bar{\eta}$ as well as ghost sources σ and $\bar{\sigma}$:

$$S_{\mathrm{C}} = -\int d^4x\,(J_\mu A^\mu + \bar{\eta}\,\psi + \bar{\psi}\,\eta + \bar{\sigma}\,c + \bar{c}\,\sigma)\,. \tag{2.2.108}$$

BRST symmetry. There is another, more economical way to arrive at Eq. (2.2.106), which is to impose **BRST invariance** of the action (Becchi, Rouet, Stora, Tyutin). A BRST transformation is defined as an infinitesimal gauge transformation (2.1.37) where the gauge parameter is a ghost field $c(x) = \sum_a c_a(x)\,\mathsf{t}_a$, i.e., where the c_a are scalar anticommuting Grassmann fields:

$$\boldsymbol{\delta}\psi = ic\,\psi\,, \qquad \boldsymbol{\delta}\bar{\psi} = -i\bar{\psi}\,c\,, \qquad \boldsymbol{\delta}A_\mu = \frac{1}{g}D_\mu c\,, \qquad \boldsymbol{\delta}F_{\mu\nu} = i\,[c,F_{\mu\nu}]\,. \tag{2.2.109}$$

We further demand this transformation to be nilpotent ($\boldsymbol{\delta}^2 = 0$), so that $\boldsymbol{\delta}$ is also Grassmann-valued and anticommutes with c. Then it is straightforward to prove that any of the relations above fixes the transformation behavior of the ghost itself, for example:

$$\boldsymbol{\delta}^2\psi = \boldsymbol{\delta}\,(ic\psi) = i(\boldsymbol{\delta}c)\,\psi - ic\,(\boldsymbol{\delta}\psi) = (i\boldsymbol{\delta}c + c^2)\,\psi \overset{!}{=} 0\,, \tag{2.2.110}$$

so we have

$$\boldsymbol{\delta}c = ic^2 = ic_a\,c_b\,\mathsf{t}_a\,\mathsf{t}_b = ic_a\,c_b\left(\tfrac{1}{2}[\mathsf{t}_a,\mathsf{t}_b] + \tfrac{1}{2}\{\mathsf{t}_a,\mathsf{t}_b\}\right) = -\tfrac{1}{2}\,f_{abc}\,c_a\,c_b\,\mathsf{t}_c \tag{2.2.111}$$

because $c_a\, c_b = -c_b\, c_a$ is antisymmetric. (Note that the Grassmann nature of the ghost fields c_a leads to weird relations such as $[c, c] = 2c^2$.) Thus, in components the BRST transformation of the ghost field becomes $\boldsymbol{\delta} c_a = -\frac{1}{2}\, f_{abc}\, c_b\, c_c$.

Applying $\boldsymbol{\delta}$ increases the ghost number (the charge corresponding to a $U(1)$ symmetry of the ghost fields) by one unit; hence, when applied to the antighost, it must produce a scalar field with ghost number zero, the so-called Nakanishi-Lautrup field: $\boldsymbol{\delta}\bar{c}_a =: -B_a/g$. Nilpotency of the antighost transformation then fixes $\boldsymbol{\delta} B = 0$. The different treatment of c and $\bar{c}$ implies that they are not conjugates of each other but truly independent fields.

Since the classical action $S[A,\, \psi,\, \bar{\psi}\,]$ is gauge invariant and BRST is a gauge transformation, it is also BRST invariant. The most general BRST-invariant action is then the sum of the classical action plus a term $S_{\rm GF} = \boldsymbol{\delta}\mathcal{O}$ which is a BRST variation itself, since in that case we have $\boldsymbol{\delta} S_{\rm GF} = \boldsymbol{\delta}^2\mathcal{O} = 0$. Adding this to the action means fixing a gauge; which gauge we get depends on $\mathcal{O}$. To recover (2.2.106), we contract the antighost with our earlier gauge-fixing condition $f[A] + \frac{\xi}{2}\, B$:

$$\begin{aligned} S_{\rm GF} &= -g\,\boldsymbol{\delta} \int d^4x\, \bar{c}_a \left(f_a[A] + \frac{\xi}{2}\, B_a \right) \\ &= \int d^4x\, B_a \left(f_a[A] + \frac{\xi}{2}\, B_a \right) + g \int d^4x \int d^4y\, \bar{c}_a(x)\, M_{ab}[A](x,y)\, c_b(y). \end{aligned} \tag{2.2.112}$$

Inserting the equations of motion for B_a, namely $f_a + \xi B_a = 0$, yields again Eq. (2.2.106); the same result follows from integrating over B_a in the path integral. Hence, imposing BRST invariance simultaneously generates gauge-fixing and ghost terms in the action.

Ward-Takahashi and Slavnov-Taylor identities. Correlation functions are not gauge invariant, but the gauge invariance of the generating functional (2.2.107) can be used to derive identities for them. As in the derivation of Dyson-Schwinger equations, a gauge transformation under the path integral is just a relabeling of fields, so Z is invariant; also the classical action is gauge invariant. The only gauge-dependent terms in (2.2.107) are then $S_{\rm GF}$ and S_C, and as in the derivation in Eq. (2.2.36), invariance of Z leads to the relation

$$\langle\, \delta S_{\rm GF} + \delta S_{\rm C}\, \rangle_J = 0\,, \tag{2.2.113}$$

which represents the generic form of a **Ward-Takahashi identity**. Then, employing Eqs. (2.2.31), (2.2.49) or (2.2.55), one can write this as a master equation for full (with sources J and derivatives $Z'[J]$), connected (J and $W'[J]$) or 1PI correlation functions (with sources φ and derivatives of the effective action, $\Gamma'[\varphi]$), see Sec. 3.1.2 for details.

In the case of non-Abelian gauge theories it is more convenient to exploit BRST invariance. Here $\boldsymbol{\delta} S_{\rm GF}$ vanishes as well and only the BRST variations of the fields in the source term remain to be evaluated: $\langle \boldsymbol{\delta} S_{\rm C} \rangle_J = 0$. In the compact notation from earlier:

$$-\,\langle \boldsymbol{\delta} S_{\rm C} \rangle_J = \int d^4x \sum_i J_i\, \langle \boldsymbol{\delta}\phi_i \rangle_J = \int d^4x \sum_i \frac{\delta\Gamma}{\delta\varphi_i}\, \langle \boldsymbol{\delta}\phi_i \rangle_J = 0\,, \tag{2.2.114}$$

where the $\boldsymbol{\delta}\phi_i$ are now nonlinear functions of the fields themselves, cf. Eq. (2.2.109). This leads to the **Slavnov-Taylor identities**.