2.3 Renormalization

We are now *almost* in a position to write down the Feynman rules of QCD. In our discussion so far we have still bypassed the problem of renormalization. The need for renormalization is related to the behavior of a theory at infinitely large momenta and in practice arises in the calculation of loop diagrams, which are usually UV-divergent. Below we will see that the problem can be dealt with by introducing a small number of renormalization constants and setting corresponding renormalization conditions, which makes all correlation functions finite. We will also see that *renormalizability* is a deep property of a QFT that can already be read off from the Lagrangian of the theory.

2.3.1 Feynman rules of QCD

Renormalization constants. A possible starting point when dealing with renormalization is to interpret all fields, masses and couplings in the Lagrangian (2.1.29) as 'bare' and unphysical, and define their renormalized versions by:

$$\psi_B = Z_\psi^{1/2}\,\psi\,, \quad A_B = Z_A^{1/2}A\,, \quad c_B = Z_c^{1/2}\,c\,, \quad m_B = Z_m\,m\,, \quad g_B = Z_g\,g\,. \tag{2.3.1}$$

The quantities without a subscript are the renormalized ones and they are related to the bare quantities by **renormalization constants**. Then the full Lagrangian of QCD including the gauge-fixing terms becomes

$$\begin{aligned}\mathcal{L}_{\text{QCD}} = {} & Z_\psi\,\bar\psi\,(i\partial\!\!\!/ - Z_m\,m)\,\psi + \frac{1}{2}A_\mu^a\left[Z_A\,(\Box\,g^{\mu\nu} - \partial^\mu\partial^\nu) + \frac{1}{\xi}\,\partial^\mu\partial^\nu\right]A_\nu^a + Z_c\,\bar c_a\,\Box\,c_a \\ & - Z_{3g}\,\frac{g}{2}\,f_{abc}\,(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)\,A_\mu^b\,A_\nu^c - Z_{4g}\,\frac{g^2}{4}\,f_{abe}\,f_{cde}\,A_a^\mu\,A_b^\nu\,A_\mu^c\,A_\nu^d \\ & + Z_\Gamma\,g\,\bar\psi A\!\!\!/\,\psi - \widetilde{Z}_\Gamma\,g f_{abc}\,(\partial_\mu\bar c_a)\,A_b^\mu\,c_c\,. \end{aligned} \tag{2.3.2}$$

The first line contains the tree-level quark, gluon and ghost propagators, the second line the three- and four-gluon interactions, and the third line the quark-gluon and ghost-gluon interaction vertices. The renormalization constants for the vertices are related to those in (2.3.1) by

$$Z_\Gamma = Z_g\,Z_A^{1/2}\,Z_\psi\,, \quad \widetilde{Z}_\Gamma = Z_g\,Z_A^{1/2}\,Z_c\,, \qquad Z_{3g} = Z_g\,Z_A^{3/2}\,, \quad Z_{4g} = Z_g^2\,Z_A^2\,. \tag{2.3.3}$$

In principle we could have different renormalization constants for each term in the Lagrangian, but the Slavnov-Taylor identities ensure that this is not the case. Thus, we have five independent renormalization constants Z_ψ, Z_A, Z_c, Z_m, Z_g, which means that at some point we will need to set five *renormalization conditions*.

Moreover, the renormalization constants also enter in the **Feynman rules** since they are derived from the Lagrangian (2.3.2). In the following we write down the Feynman rules for the renormalized propagators and 1PI vertices of QCD.

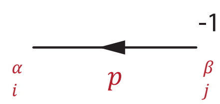
Before doing so, we note that one could equivalently introduce renormalization constants in the language of **counterterms**:

$$Z_\psi = 1 + \delta Z_\psi\,, \qquad Z_\psi\, Z_m\, m = m + \delta m\,, \qquad Z_g\, g = g + \delta g\,, \quad \dots \tag{2.3.4}$$

In this way we would split the Lagrangian into two pieces, where the first has the original form but with renormalized fields and without renormalization constants, and the second contains the counterterms which generate new propagators and vertices with new Feynman rules. We will not follow this strategy here and instead absorb the renormalization constants directly in the Feynman rules. Also, note that the renormalization constants in the literature usually go by different names:

$$Z_\psi = Z_2\,, \quad Z_A = Z_3\,, \quad Z_c = \widetilde{Z}_3\,, \quad Z_\Gamma = Z_{1f}\,, \quad \widetilde{Z}_\Gamma = \widetilde{Z}_1\,, \quad Z_{3g} = Z_1\,, \quad Z_{4g} = Z_4\,. \tag{2.3.5}$$

Quark propagator. The quark propagator is a Dirac matrix with indices α and β, it depends on one momentum p^μ, and it is a diagonal matrix δ_{ij} with $i, j = 1, 2, 3$ in color space. (We ignore flavor since it merely amounts to replicating terms in the Lagrangian.) Since we count the spin indices from the top of the arrow, i.e. from left to right, we also let the momentum flow from right to left. Writing $Z_m\, m = m_B$, where m is the renormalized current-quark mass, the inverse tree-level quark propagator from the Lagrangian is then given by (we suppress the color indices on the l.h.s.)



$$\begin{aligned} S_0^{-1}(p) &= -iZ_\psi\left(\not p - m_B\right)\delta_{ij}\,, \\ S_0(p) &= \frac{i}{Z_\psi}\,\frac{\not p + m_B}{p^2 - m_B^2 + i\epsilon}\,\delta_{ij}\,. \end{aligned} \tag{2.3.6}$$

Can we also write down a ‘Feynman rule’ for the *full* propagator $S(p)$? In general, any n-point correlation function can be expanded in a tensor basis

$$G^{\mu\nu\dots}_{\alpha\beta\dots}(p_1,\dots p_n) = \sum_{i=1}^{N} f_i(p_1^2, p_2^2, \dots)\,\tau_i(p_1,\dots p_n)^{\mu\nu\dots}_{\alpha\beta\dots}\,, \tag{2.3.7}$$

where the τ_i are Lorentz-covariant tensors that inherit the Lorentz and Dirac structure of G. The f_i are Lorentz-invariant **dressing functions** (‘form factors’), which depend on all possible Lorentz-invariant momentum variables — they contain the physical information encoded in the correlation function. Like G itself, the basis elements transform under finite-dimensional representations of the Lorentz group,

$$\tau_i(p_1',\dots p_n')^{\mu\nu\dots}_{\alpha\beta\dots} = \Lambda^\mu{}_{\mu'}\,\Lambda^\nu{}_{\nu'}\cdots D_{\alpha\alpha'}(\Lambda)\,\tau(p_1,\dots p_n)^{\mu'\nu'\dots}_{\alpha'\beta'\dots}\,D^{-1}_{\beta'\beta}(\Lambda)\,, \tag{2.3.8}$$

where Λ is the Lorentz transformation and $D(\Lambda)$ its spinor representation matrix (see Appendix B). The same formula holds for parity transformations if $D(\Lambda)$ is replaced by γ^0. In practice, this means that the tensors are constructed by combining

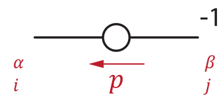
$$g^{\mu\nu}\,, \quad \varepsilon^{\mu\nu\alpha\beta}\,, \quad \mathbb{1}\,, \quad \gamma^\mu\,, \quad \gamma_5\gamma^\mu\,, \quad \sigma^{\mu\nu}\,, \quad \gamma_5\sigma^{\mu\nu} \tag{2.3.9}$$

with the four-momenta in the system.

The quark propagator depends on only one momentum,

$$S_{\alpha\beta}(p) = \sum_{i=1}^{2} f_i(p^2)\,\tau_i(p)_{\alpha\beta}\,, \tag{2.3.10}$$

and from (2.3.9) we can only construct the two tensors $\mathbb{1}$ and $\not{p}$ since those with γ_5 would have the wrong parity. Thus, the full quark propagator can be written as



$$\begin{aligned} S^{-1}(p) &= -iA(p^2)\left(\not{p} - M(p^2)\right)\delta_{ij}\,, \\ S(p) &= \frac{i}{A(p^2)}\,\frac{\not{p}+M(p^2)}{p^2-M(p^2)^2+i\epsilon}\,\delta_{ij}\,. \end{aligned} \tag{2.3.11}$$

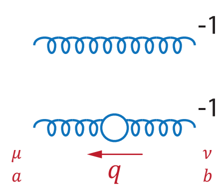
Here we defined the **quark mass function** $M(p^2)$, and the inverse of $A(p^2)$ is called the quark 'wave-function renormalization' $Z_f(p^2) = 1/A(p^2)$. If we knew these two functions for all $p^2 \in \mathbb{C}$ (recall the discussion around Fig. 2.5), we would know the full quark propagator in QCD. To project out the dressing functions, we take Dirac traces:

$$M(p^2)A(p^2) = -\frac{i}{4}\,\mathrm{Tr}\left\{S^{-1}(p)\right\}, \qquad A(p^2) = \frac{i}{4p^2}\,\mathrm{Tr}\left\{\not{p}\,S^{-1}(p)\right\}. \tag{2.3.12}$$

Gluon propagator. The gluon propagator depends on two Lorentz indices μ, ν and one momentum q; from this we can only form the two tensors $g^{\mu\nu}$ and $q^\mu q^\nu$. It is useful to define the transverse and longitudinal projectors as their linear combinations:

$$T_q^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\,, \qquad L_q^{\mu\nu} = \frac{q^\mu q^\nu}{q^2}\,. \tag{2.3.13}$$

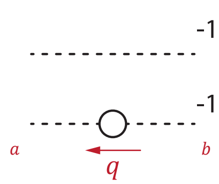
Then the Feynman rules for the tree-level and full gluon propagator are



$$\begin{aligned} (D_0^{-1})^{\mu\nu}(q) &= iq^2\left[Z_A\,T_q^{\mu\nu} + \frac{1}{\xi}\,L_q^{\mu\nu}\right]\delta_{ab}\,, \\ (D^{-1})^{\mu\nu}(q) &= iq^2\left[\frac{1}{Z(q^2)}\,T_q^{\mu\nu} + \frac{1}{\xi}\,L_q^{\mu\nu}\right]\delta_{ab}\,, \\ D^{\mu\nu}(q) &= -\frac{i}{q^2+i\epsilon}\left[Z(q^2)\,T_q^{\mu\nu} + \xi\,L_q^{\mu\nu}\right]\delta_{ab}\,, \end{aligned} \tag{2.3.14}$$

where $Z(q^2)$ is the **gluon dressing function**. (In principle the longitudinal part could also pick up a dressing, but the Slavnov-Taylor identity prevents this and ensures that the longitudinal part remains undressed.) The gluon is color-diagonal with $a, b = 1 \dots 8$.

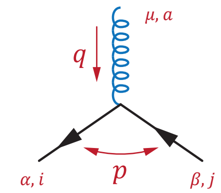
Ghost propagator. The ghost propagator is scalar and thus the simplest case, since it has no tensor structure and there is only one ghost dressing function:



$$\begin{aligned} D_{G,0}^{-1}(q) &= iq^2\,Z_c\,\delta_{ab}\,, \\ D_G^{-1}(q) &= iq^2\,G(q^2)^{-1}\,\delta_{ab}\,, \\ D_G(q) &= -\frac{i}{q^2+i\epsilon}\,G(q^2)\,\delta_{ab}\,. \end{aligned} \tag{2.3.15}$$

Note that if we had not absorbed the minus sign into the antighost field in the third line of Eq. (2.2.106), the Feynman rules for the ghost propagator and ghost-gluon vertex would come with additional minus signs.

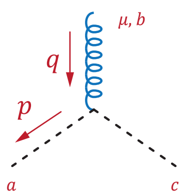
Quark-gluon vertex. The Feynman rules for the quark-gluon vertex are



$$\begin{aligned} \Gamma^\mu_0 &= ig\,(\mathsf{t}_a)_{ij}\, Z_\Gamma\, \gamma^\mu\,, \\ \Gamma^\mu(p,q) &= ig\,(\mathsf{t}_a)_{ij} \sum_{i=1}^{12} f_i(p^2,q^2,p\cdot q)\,\tau_i^\mu(p,q)\,. \end{aligned} \tag{2.3.16}$$

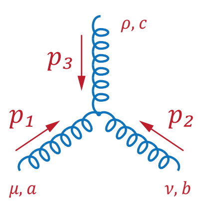
The full vertex becomes rather complicated since it depends on two independent momenta p and q. This leads to 12 possible tensors that are allowed by Lorentz covariance: γ^μ, p^μ, q^μ, $[\gamma^\mu, \not{p}]$, ..., and the dressing functions depend on the three Lorentz invariants p^2, q^2 and $p\cdot q$. Since the vertex has a charge-conjugation symmetry, it is convenient to identify p with the average momentum between the incoming and outgoing quarks because this makes the symmetry manifest in the dressing functions (the dependence on $p\cdot q$ is then either even or odd).

Ghost-gluon vertex. The ghost-gluon vertex has no Dirac structure and therefore only two tensors p^μ and q^μ. In this case the tree-level vertex depends on the *outgoing* momentum p^μ because in the Lagrangian (2.3.2) the derivative acts on $\bar{c}_a$ (i.e., the ghost and antighost fields are not related by charge conjugation):



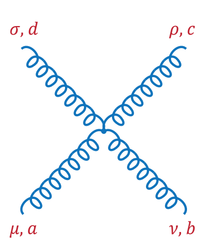
$$\begin{aligned} \Gamma^\mu_{\text{gh},0}(p) &= g f_{abc}\, \widetilde{Z}_\Gamma\, p^\mu\,, \\ \Gamma^\mu_{\text{gh}}(p,q) &= g f_{abc} \sum_{i=1}^{2} \widetilde{f}_i(p^2,q^2,p\cdot q)\,\tau_i^\mu(p,q)\,. \end{aligned} \tag{2.3.17}$$

Three-gluon vertex. Here things get a bit more cumbersome since the full vertex has 14 possible Lorentz tensors. The tree-level vertex (with $p_1+p_2+p_3=0$) reads:



$$\begin{aligned} \Gamma^{\mu\nu\rho}_{3g,0}(p_1,p_2,p_3) = g f_{abc}\, Z_{3g}\, \big[&(p_1-p_2)^\rho\, g^{\mu\nu} \\ &+(p_2-p_3)^\mu\, g^{\nu\rho} + (p_3-p_1)^\nu\, g^{\rho\mu}\big]\,. \end{aligned} \tag{2.3.18}$$

Four-gluon vertex. In this case things get *really* cumbersome: The full vertex has 136 linearly independent Lorentz tensors and five color structures. The tree-level vertex is momentum-independent:



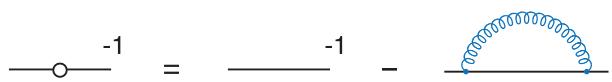
$$\begin{aligned} \Gamma^{\mu\nu\rho\sigma}_{4g,0} = -ig^2 Z_{4g}\Big[\; &f_{abe} f_{cde}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}) \\ +\; &f_{ace} f_{bde}\,(g^{\mu\nu}g^{\rho\sigma} - g^{\nu\rho}g^{\mu\sigma}) \\ +\; &f_{ade} f_{cbe}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma})\Big]\,. \end{aligned} \tag{2.3.19}$$

One-loop perturbation theory. With the Feynman rules at hand, we are ready to set up perturbation theory. To avoid redundancies, we will do this for the 1PI correlation functions, i.e. we set up the perturbative expansion for the *inverse* propagators, in the same way as we wrote the Dyson-Schwinger equations in Eq. (2.2.62) and thereafter. Then we only need to work out the self-energy diagrams, whereas the expansion for the propagator is easily obtained from Eq. (2.2.63) if needed.

The DSE for the **quark propagator** has the generic form

$$S^{-1}(p) = S_0^{-1}(p) - i\Sigma(p)\,, \tag{2.3.20}$$

where the quark self-energy $\Sigma(p)$ contains only one diagram at one-loop order:



If we write $\Sigma(p) = \Sigma_A(p^2)\,\not{p} - \Sigma_M(p^2)$ and insert Eqs. (2.3.6) and (2.3.11), we read off the relations for the two scalar dressing functions:

$$\begin{aligned} A(p^2) &= Z_\psi + \Sigma_A(p^2)\,, \\ M(p^2)A(p^2) &= Z_\psi Z_m\, m + \Sigma_M(p^2)\,. \end{aligned} \tag{2.3.21}$$

We will later see that the renormalization constants have the structure $Z = 1 + \mathcal{O}(g^2)$, and since the self-energy comes with a factor g^2, the mass function up to $\mathcal{O}(g^2)$ is

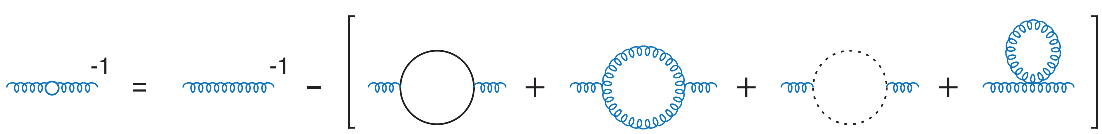
$$M(p^2) = Z_m\, m + \Sigma_M(p^2) - m\,\Sigma_A(p^2)\,. \tag{2.3.22}$$

We will work out the self-energy explicitly in Sec. 2.3.2.

The DSE for the **gluon propagator** is given by

$$(D^{-1})^{\mu\nu}(q) = (D_0^{-1})^{\mu\nu}(q) - i\Pi^{\mu\nu}(q)\,, \tag{2.3.23}$$

where $\Pi^{\mu\nu}(q)$ is the gluon vacuum polarization. At one-loop order $\mathcal{O}(g^2)$ it consists of a quark loop, a gluon loop, a ghost loop, and a tadpole diagram:



We can split the vacuum polarization into two terms,

$$\Pi^{\mu\nu}(q) = \Pi(q^2)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) + \widetilde{\Pi}(q^2)\, g^{\mu\nu} = \Pi(q^2)\, q^2\, T_q^{\mu\nu}\,, \tag{2.3.24}$$

where only the first survives because the Slavnov-Taylor identity entails $q_\mu \Pi^{\mu\nu}(q) = 0$ and therefore $\widetilde{\Pi}(q^2) = 0$. (The one-loop result for $\widetilde{\Pi}(q^2)$ indeed vanishes in dimensional regularization, but it is non-zero for a cutoff regulator which breaks gauge invariance.) Inserting Eq. (2.3.14) into the DSE, we see that there are no loop corrections for the longitudinal part, which is also why no renormalization is required for this term. The equation then simply becomes

$$Z(q^2)^{-1} = Z_A - \Pi(q^2)\,. \tag{2.3.25}$$

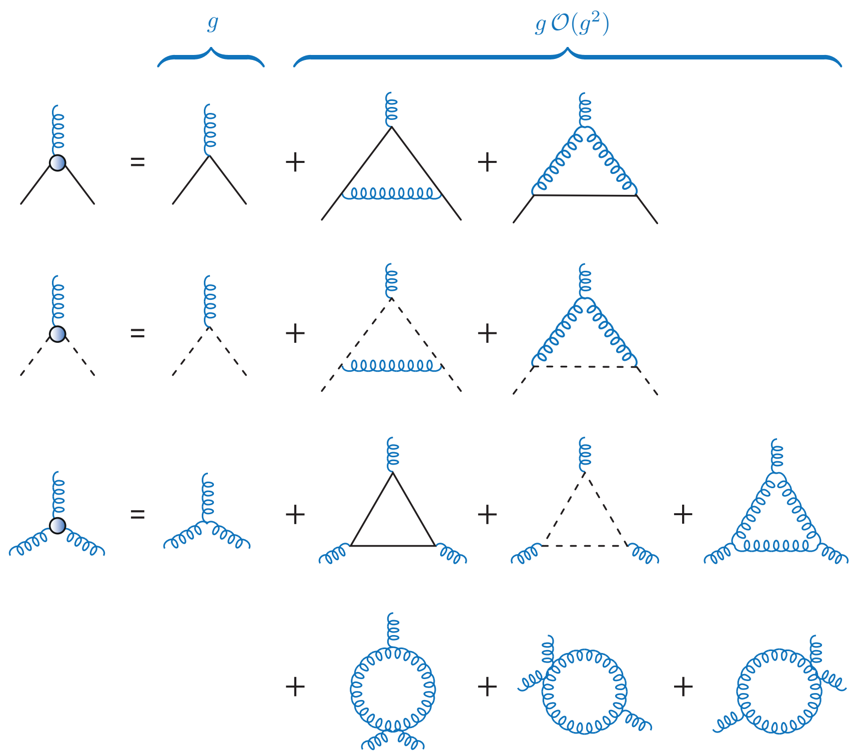
In the analogous case of QED, after renormalization $\Pi(q^2)$ becomes constant for $q^2 \to 0$, which means that the photon propagator has a massless $1/q^2$ pole and the photon remains massless also with interactions. In QCD, this is still what happens in perturbation theory but it may no longer be true non-perturbatively. Early ideas suggested a $1/q^4$ pole for the gluon 'propagator' $Z(q^2)/q^2$ since this would signal confinement: if one connects a quark and antiquark by a gluon, the three-dimensional Fourier transform of $1/|\boldsymbol{q}|^4$ leads to a potential $\propto |\boldsymbol{r}|$ in coordinate space simply by dimensional counting. Nowadays evidence from non-perturbative (lattice and functional) calculations in Landau gauge suggests that this is not what happens and that $Z(q^2)$ instead *vanishes* at $q^2 = 0$, either with a power q^2 (**'massive'** or **'decoupling'** scenario) or higher (**'scaling'** scenario). As a result, $Z(q^2)/q^2$ becomes constant or even has a turnover in the infrared. Vice versa, $Z(q^2)^{-1}$ and therefore $\Pi(q^2)$ must be singular at $q^2 \to 0$, but the origin of this singularity is still under debate. Moreover, in the scaling scenario the infrared exponents for any quark-antiquark interaction diagram still match to produce a $1/q^4$ behavior (e.g., for the combination of a gluon propagator and two quark-gluon vertices)[3], whereas in the massive scenario (which is supported by lattice calculations) this is not the case.

The DSE for the **ghost propagator** reads

$$D_G^{-1}(q) = D_{G,0}^{-1}(q) - iq^2\,\Sigma_G(q^2) \qquad \Rightarrow \qquad G(q^2)^{-1} = Z_c - \Sigma_G(q^2)\,, \qquad (2.3.26)$$

where the perturbative expansion of the self-energy is analogous to the quark:

Finally, the one-loop expressions of the quark-gluon, ghost-gluon and three-gluon **vertices** have the form (note that a factor g is implicit in the vertices):



[3] R. Alkofer, C. S. Fischer, F. J. Llanes-Estrada, Mod. Phys. Lett. A 23 (2008) 1105, hep-ph/0607293.

2.3.2 Regularization and renormalization

In practice the diagrams we just drew are UV-divergent. The first step in dealing with this problem is regularization, which means to *isolate* the divergences. In the second step we *remove* the divergences; as we will see, there is a systematic procedure behind it, namely renormalization.

Feynman parameters. But first of all we must bring the integrals into a manageable form. To do so, we use the 'Feynman trick', where we write the quantity $1/(A_1 \dots A_n)$ as an integral over Feynman paramaters $x_1, \dots x_n$:

$$\frac{1}{A_1 \dots A_n} = \underbrace{\int dx_1 \dots dx_n \, \delta\left(\sum_{i=1}^n x_i - 1\right)}_{d\Omega_n} \frac{(n-1)!}{\left(\sum_{i=1}^n x_i A_i\right)^n} \,. \tag{2.3.27}$$

This is the generalization of the identity

$$\int_0^1 dx \int_0^1 dy \, \frac{\delta(x+y-1)}{(xA+yB)^2} = \int_0^1 dx \, \frac{1}{[xA+(1-x)B]^2} = -\frac{1}{A-B} \, \frac{1}{xA+(1-x)B}\bigg|_0^1 = \frac{1}{AB} \,. \tag{2.3.28}$$

The integral measure $d\Omega_n$ for $n=2$ and $n=3$ is

$$\begin{aligned} &\int d\Omega_2 \dots = \int_0^1 dx \dots \bigg|_{y=1-x} \,, \\ &\int d\Omega_3 \dots = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \dots \bigg|_{x_3 = 1-x_1-x_2} = \frac{1}{2} \int_0^1 da \int_{-a}^{a} db \dots \end{aligned} \tag{2.3.29}$$

Here we set $a = x_1 + x_2 = 1 - x_3$ and $b = x_1 - x_2$, which is convenient since the integral over b is antisymmetric and thus only even terms in b survive. Similar expressions hold for $d\Omega_4$, $d\Omega_5$ etc.

Now consider a generic one-loop diagram L_n which has n propagators in the loop. If we write $A_i = (k+p_i)^2 - m_i^2 + i\epsilon$, where k is the loop momentum and the p_i are external momenta, its structure will always be the same irrespective of the theory:

$$L_n = \int \frac{d^4k}{(2\pi)^4} \frac{(\dots)}{\prod_{i=1}^n A_i} = (n-1)! \int d\Omega_n \int \frac{d^4k}{(2\pi)^4} \frac{(\dots)}{\left(\sum_{i=1}^n x_i A_i\right)^n} \,. \tag{2.3.30}$$

The numerator $(\dots)$ can have Lorentz and Dirac indices and in general it also depends on k and p_i. If we define a new loop momentum l by

$$l = k + \sum_{i=1}^n x_i \, p_i \,, \tag{2.3.31}$$

then with $\sum_i x_i = 1$ it is easy to show that

$$\sum_{i=1}^n x_i \, A_i = l^2 - \Delta + i\epsilon \,, \qquad \Delta = \left(\sum_i x_i \, p_i\right)^2 - \sum_i x_i \left(p_i^2 - m_i^2\right) , \tag{2.3.32}$$

where Δ does not depend on l but only on the external momenta p_i and the Feynman parameters x_i. Thus we obtain

$$L_n = (n-1)! \int d\Omega_n \int \frac{d^4 l}{(2\pi)^4} \frac{(\dots)}{(l^2 - \Delta + i\epsilon)^n} \,. \tag{2.3.33}$$

Finally, we perform a Wick rotation (see Appendix C)

$$l_4 = i l_0 \quad \Rightarrow \quad l^2 = -l_4^2 - \boldsymbol{l}^2 = -l_E^2 \,, \qquad \int d^4 l = -i \int d^3 l \int\limits_{\infty}^{-\infty} dl_4 = i \int d^4 l_E \tag{2.3.34}$$

to arrive at the Euclidean integral

$$L_n = i(-1)^n (n-1)! \int d\Omega_n \int \frac{d^4 l_E}{(2\pi)^4} \frac{(\dots)}{(l_E^2 + \Delta)^n} \,. \tag{2.3.35}$$

Usually the hardest part is to work out the numerator, where we also have to express k in terms of l and the p_i through Eq. (2.3.31). In doing so, it will depend on powers of the loop momentum l^μ. What helps is that integrals over odd powers vanish by symmetry (replace $l^\mu \to -l^\mu$), e.g.

$$\int \frac{d^4 l_E}{(2\pi)^4} \frac{l^\mu}{(l_E^2 + \Delta)^2} = 0 \,, \tag{2.3.36}$$

whereas even powers can always be reduced to integrals of the form

$$I_{nm} = \int \frac{d^4 l_E}{(2\pi)^4} \frac{(l_E^2)^m}{(l_E^2 + \Delta)^n} \,. \tag{2.3.37}$$

For example,

$$\int \frac{d^4 l_E}{(2\pi)^4} \frac{l^\mu l^\nu}{(l_E^2 + \Delta)^2} = -\frac{1}{4} g^{\mu\nu} \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2}{(l_E^2 + \Delta)^2} \tag{2.3.38}$$

because for $\mu \neq \nu$ the integral vanishes again by symmetry, whereas for $\mu = \nu$ it must be proportional to $g^{\mu\nu}$ by Lorentz invariance. The prefactors are then determined by contracting the indices on both sides, using $l^2 = -l_E^2$ and $\delta^\mu_\mu = 4$ (note that in d dimensions one has $\delta^\mu_\mu = d$, so the prefactor on the r.h.s. becomes $-1/d$).

As a consequence, the numerator under the integral in Eq. (2.3.35) can be written as $(\dots) = \sum_m (\dots)_m \, (l_E^2)^m$, and the integral becomes

$$L_n = i(-1)^n (n-1)! \int d\Omega_n \sum_m (\dots)_m \, I_{nm} \,. \tag{2.3.39}$$

Dimensional regularization. The remaining task is to work out the integrals I_{nm}, which are divergent for $n - m \leq 2$. The idea of **regularization** is to isolate the divergent pieces and write the expressions as a sum of finite and divergent terms. In the following we use dimensional regularization, where we generalize the $d^4 l$ integral to d dimensions:

$$I_{nm}^{(d)} = \frac{1}{M^{d-4}} \int \frac{d^d l_E}{(2\pi)^d} \frac{(l_E^2)^m}{(l_E^2 + \Delta)^n} \,. \tag{2.3.40}$$

To preserve the mass dimension, we put an (arbitrary) mass scale M in front of the integral. This seemingly innocuous operation has profound consequences, namely: *Regularization always introduces a scale.* When splitting the integrals into finite and divergent pieces, the finite terms still depend on this scale, which cannot be removed.

There are many different ways to regularize the theory: instead of dimensional regularization, which is convenient for perturbative calculations, we could also

- introduce a hard momentum cutoff $\int_0^\infty dl_E^2 \to \int_0^{\Lambda^2} dl_E^2$, which unfortunately breaks gauge invariance;
- use Pauli-Villars regularization, where we subtract each propagator by another propagator with a large mass M,
- or use a lattice regularization, where we discretize spacetime and introduce a lattice spacing a.

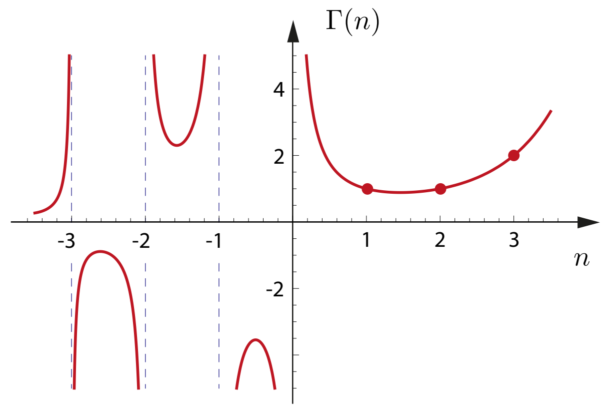
In all these cases we end up with an arbitrary mass scale in the theory: the mass M in dimensional or Pauli-Villars regularization, the cutoff Λ, or the inverse lattice spacing $1/a$. Later we will see that we can trade the dependence on this scale for a dependence on an arbitrary renormalization point. Even for the massless QCD Lagrangian, which has no intrinsic scale and is therefore scale invariant, regularization introduces a scale. (And fortunately so, because if we were to compute the hadron spectrum of massless QCD, we would otherwise expect all hadrons to be massless since nothing sets the scale.) This is also called **anomalous breaking of scale invariance**, since an anomaly is a symmetry of the classical action that is broken at the quantum level.

Moving on with dimensional regularization, we do not repeat the calculation for the integral (2.3.40) (which can be found in QFT textbooks) but only quote its result:

$$\begin{aligned} I_{nm}^{(d)} &= \frac{1}{M^{d-4}}\,\frac{1}{(4\pi)^{d/2}}\,\frac{1}{\Gamma(n)}\,\frac{1}{\Delta^{n-m-d/2}}\,\frac{\Gamma(\frac{d}{2}+m)}{\Gamma(\frac{d}{2})}\,\Gamma(n-m-\tfrac{d}{2}) \\ &\stackrel{d=4-\varepsilon}{=} \frac{1}{(4\pi)^2}\,\frac{\Gamma(m+2-\frac{\varepsilon}{2})}{\Gamma(n)\,\Gamma(2-\frac{\varepsilon}{2})}\,\frac{1}{\Delta^{n-m-2}}\left(\frac{4\pi M^2}{\Delta}\right)^{\varepsilon/2}\Gamma(n-m-2+\tfrac{\varepsilon}{2}). \end{aligned} \tag{2.3.41}$$

$\Gamma(n)$ is the Gamma function, which provides an analytic continuation of the result for arbitrary values of d. It has the properties

- $\Gamma(n) = \int_0^\infty dx\, x^{n-1} e^{-x}$,
- $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}_+$,
- $\Gamma(n)$ has poles at $n = 0, -1, -2, \dots$
- $\Gamma(n+1) = n\,\Gamma(n)$,
- $\Gamma'(1) = -\gamma = -0.5772\dots$ is the Euler-Mascheroni constant.



For $\varepsilon \to 0$ and thus $d \to 4$, one can see that (2.3.41) is divergent for $n - m - 2 \leq 0$. In this case we can use

$$\Gamma(\tfrac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)\,, \qquad x^{\varepsilon/2} = 1 + \frac{\varepsilon}{2}\ln x + \mathcal{O}(\varepsilon^2) \tag{2.3.42}$$

to obtain the convergent integrals

$$\{I_{30}\,,\ I_{40}\,,\ I_{41}\,,\dots\} = \frac{1}{(4\pi)^2}\left\{\frac{1}{2\Delta}\,,\ \frac{1}{6\Delta^2}\,,\ \frac{1}{3\Delta}\,,\ \dots\right\}. \tag{2.3.43}$$

The divergent integrals are given by

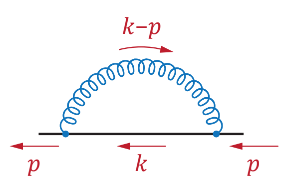
$$\{I_{20}\,,\ I_{31}\,,\ I_{42}\,,\dots\} = \frac{1}{(4\pi)^2}\left\{D\,,\ D - \tfrac{1}{2}\,,\ D - \tfrac{5}{6}\,,\ \dots\right\} \tag{2.3.44}$$

with

$$D = \frac{2}{\varepsilon} - \gamma + \ln\frac{4\pi M^2}{\Delta} + \mathcal{O}(\varepsilon)\,. \tag{2.3.45}$$

Be careful with the limit $\varepsilon \to 0$ for the divergent terms: also $\mathcal{O}(\varepsilon)$ terms must be kept in the calculation since they combine with the $1/\varepsilon$ term to give a finite contribution. In conclusion, we have managed to split the integrals into divergent pieces, where the UV divergences appear in the form of $1/\varepsilon$ terms, and finite pieces which depend on the arbitrary mass scale M.

Quark self-energy. Let us work out a concrete example, namely the quark self-energy from Eq. (2.3.20):



Using the Feynman rules, it reads explicitly:

$$\begin{aligned} i\Sigma(p) &= \int\frac{d^4k}{(2\pi)^4}\,(ig\gamma_\mu)\,S_0(k)\,(ig\gamma_\nu)\,D_0^{\mu\nu}(k-p)\Big(\sum_a \mathsf{t}_a\,\mathsf{t}_a\Big)_{ij} \\ &= -g^2\,C_F\,\delta_{ij}\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu\,(\not{k}+m)\,\gamma_\mu}{[(k-p)^2+i\epsilon][k^2-m^2+i\epsilon]}\,. \end{aligned} \tag{2.3.46}$$

Here we employed the gluon propagator in Feynman gauge ($\xi = 1$), the color factor is $C_F = (N_c^2-1)/(2N_c)$, and we ignored the renormalization constants multiplying the self-energy since they have the structure $Z = 1 + \mathcal{O}(g^2)$ and will thus only contribute to higher orders in perturbation theory.

The integral is of the form (2.3.30) with $p_1 = -p$, $m_1 = 0$, $p_2 = 0$ and $m_2 = m$. Therefore, we have

$$\begin{aligned} l &= k + \sum_i x_i\,p_i = k - xp\,, \\ \Delta &= x^2\,p^2 - xp^2 - (1-x)(-m^2) = (1-x)(m^2 - xp^2)\,. \end{aligned} \tag{2.3.47}$$

Denoting the denominator by $\mathcal{N} = \gamma^\mu \,(\not{k} + m)\,\gamma_\mu$ and removing the unit matrix δ_{ij} in color space, we can immediately use the result (2.3.35):

$$i\Sigma(p) = -g^2\, C_F\, \frac{i}{M^{d-4}} \int_0^1 dx \int \frac{d^d l_E}{(2\pi)^d}\, \frac{\mathcal{N}}{(l_E^2 + \Delta)^2}\,, \tag{2.3.48}$$

which we already generalized to d dimensions.

To work out the numerator, use $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\gamma^\mu\gamma_\mu = \delta^\mu_\mu = d$ in d dimensions. This gives

$$\begin{aligned} \mathcal{N} = \gamma^\mu\,(\not{k} + m)\,\gamma_\mu &= -\gamma^\mu\gamma_\mu\,\not{k} + 2\not{k} + m\gamma^\mu\gamma_\mu \\ &= (2-d)\,\not{k} + md \\ &= (2-d)\,\not{l} + (2-d)\,x\not{p} + md\,. \end{aligned} \tag{2.3.49}$$

The first term in the last line is odd in l^μ, so it vanishes after integration according to Eq. (2.3.36), whereas the remainder is independent of the loop momentum and can be pulled out of the integral. We obtain

$$i\Sigma(p) = -ig^2\, C_F \int_0^1 dx\, I_{20}^{(d)} \left[(2-d)\,x\not{p} + md\right], \tag{2.3.50}$$

and if we split the self-energy into $\Sigma(p) = \Sigma_A(p^2)\,\not{p} - \Sigma_M(p^2)$ we can read off the scalar expressions:

$$\begin{aligned} \Sigma_A(p^2) &= g^2\, C_F\,(d-2) \int dx\, x\, I_{20}^{(d)}\,, \\ \Sigma_M(p^2) &= g^2\, C_F\, md \int dx\, I_{20}^{(d)}\,. \end{aligned} \tag{2.3.51}$$

Setting $d = 4 - \varepsilon$ and taking $\varepsilon \to 0$, with $I_{20} = D/(4\pi)^2$ and $\alpha = g^2/(4\pi)$, we finally arrive at

$$\begin{aligned} \Sigma_A(p^2) &= \frac{\alpha}{2\pi}\, C_F \int dx\, x \left(\frac{2}{\varepsilon} - \gamma + \ln\frac{4\pi M^2}{\Delta} - 1\right), \\ \Sigma_M(p^2) &= \frac{\alpha m}{\pi}\, C_F \int dx \left(\frac{2}{\varepsilon} - \gamma + \ln\frac{4\pi M^2}{\Delta} - \frac{1}{2}\right). \end{aligned} \tag{2.3.52}$$

In conclusion, we have split the quark self-energy into divergent and finite pieces. But what are we supposed to do with the divergences — throw them away? How would that make any sense? Surprisingly enough, this is indeed what eventually has to happen, but there is a deeper underlying reason which can be understood in the course of renormalization.

Renormalization. The basic idea is the following and can be motivated from QED. There, the full fermion propagator should have a pole at $p^2 = m^2$, where it returns to a free propagator but with the *physical* mass m. Thus we could impose

$$S(p) \xrightarrow{p^2=m^2} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} \qquad \Rightarrow \qquad \begin{array}{r} A(p^2=m^2) \stackrel{!}{=} 1 \\ M(p^2=m^2) \stackrel{!}{=} m\,. \end{array} \tag{2.3.53}$$

These are two conditions, where one fixes the pole position and the other the residue of the propagator. They correspond to an **onshell renormalization**; likewise, we would demand that the photon dressing function becomes $Z(q^2=0)=1$ at the onshell point.

In QCD it would not make much sense to impose such conditions, since there are no free quarks and gluons due to confinement. Fortunately, it turns out that these renormalization conditions are arbitrary and thus we can generalize them to an arbitrary **renormalization point** μ:

$$\begin{aligned} &S(p) \xrightarrow{p^2=\mu^2} \left.\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\right|_{p^2=\mu^2} \qquad \Rightarrow \qquad \begin{array}{r} A(p^2=\mu^2) \stackrel{!}{=} 1\,, \\ M(p^2=\mu^2) \stackrel{!}{=} m\,, \end{array} \\ &Z(q^2=\mu^2) \stackrel{!}{=} 1\,, \qquad G(q^2=\mu^2) \stackrel{!}{=} 1\,, \qquad \Gamma^\mu_{\text{gh}}(p^2=\mu^2) \stackrel{!}{=} g f_{abc}\, p^\mu\,. \end{aligned} \tag{2.3.54}$$

Here we imposed five conditions, four for the quark, gluon and ghost dressing functions and one for the ghost-gluon vertex. (We could have chosen any other vertex, and in fact we could have even chosen different renormalization points for each correlation function, but let's keep matters simple.)

The effect is that these five conditions determine the five renormalization constants in Eq. (2.3.1). For example, for the quark propagator we obtain according to (2.3.21):

$$\begin{array}{rl} A(\mu^2) = Z_\psi + \Sigma_A(\mu^2) & \stackrel{!}{=} 1 \\ M(\mu^2)A(\mu^2) = Z_\psi Z_m\, m + \Sigma_M(\mu^2) & \stackrel{!}{=} m \end{array} \quad \Rightarrow \quad \begin{array}{rl} Z_\psi &= 1 - \Sigma_A(\mu^2)\,, \\ Z_\psi Z_m &= 1 - \dfrac{\Sigma_M(\mu^2)}{m}\,. \end{array} \tag{2.3.55}$$

This is, by the way, also the reason why we could set the renormalization constants attached to the one-loop quark self-energy in Eq. (2.3.46) to 1, since the remaining contributions would only enter at higher loop orders. As a result, the renormalized dressing functions are finite because the $1/\varepsilon$ divergences drop out:

$$\begin{aligned} A(p^2) &= 1 + \Sigma_A(p^2) - \Sigma_A(\mu^2) = 1 + \frac{\alpha}{2\pi}\, C_F \int dx\, x\, \ln\frac{m^2 - x\mu^2}{m^2 - xp^2}\,, \\ M(p^2)A(p^2) &= m + \Sigma_M(p^2) - \Sigma_M(\mu^2) = m + \frac{\alpha m}{\pi}\, C_F \int dx\, \ln\frac{m^2 - x\mu^2}{m^2 - xp^2}\,. \end{aligned} \tag{2.3.56}$$

The resulting mass function up to $\mathcal{O}(\alpha)$ is

$$M(p^2) = m\left[1 + \frac{\alpha}{\pi}\, C_F \int dx \left(1 - \frac{x}{2}\right) \ln\frac{m^2 - x\mu^2}{m^2 - xp^2}\right] + \mathcal{O}(\alpha^2)\,, \tag{2.3.57}$$

which for p^2, $\mu^2 \gg m^2$ becomes (we will return to this at the end of Sec. 2.3.3)

$$M(p^2) \approx m\left[1 - \frac{3\alpha}{4\pi}\, C_F\, \ln\frac{p^2}{\mu^2}\right]. \tag{2.3.58}$$

We could repeat the procedure to determine the one-loop results for the remaining propagators and vertices and in all cases the $1/\varepsilon$ divergences would drop out as well. As a result, imposing five renormalization conditions determines the five renormalization constants Z_i and removes *all* divergences from the theory (we will better see how this works below). The resulting correlation functions are finite but depend on the arbitrary renormalization point μ, which replaces the dependence on the arbitrary mass scale M. The renormalization constants Z_i are still divergent since they absorb the $1/\varepsilon$ terms, but they drop out in all observables that can be calculated from the theory.

In this way, the mass $m(\mu)$ is a parameter of the theory which has to be taken from experiment. In QED we could set the physical mass of the electron by onshell renormalization ($p^2 = m^2$), because this is where the electron propagator has a pole. In QCD, the current-quark masses must be specified at some suitable renormalization scale where theory predictions can be compared to experiment. This scale should also be spacelike ($\mu^2 < 0$) to avoid branch-point singularities that appear in the loop diagrams. High-energy scattering experiments with hadrons probe quarks and gluons at large spacelike momenta, which is also where the QCD coupling is small and perturbation theory applicable.

The coupling $g(\mu)$, on the other hand, is not truly a parameter but sets the scale: so far we have been working in arbitrary units, but to connect to GeV units we must set the coupling $\alpha(\mu^2)$ at a given momentum scale. Different values of $\alpha(\mu^2)$ then merely rescale the system, which means that the *running* of the coupling $\alpha(\mu^2)$ is an inherent property of the theory itself. Therefore, the parameters of QCD are a scale, where the coupling takes a specific value, and the current-quark masses at that scale — these must be taken from experiment.

The choice of a **renormalization scheme** reflects the arbitrariness in the specification of $m(\mu)$ and $g(\mu)$:

- Imposing *overall* renormalization conditions of the form (2.3.54) defines a momentum subtraction **(MOM)** scheme. This is convenient for nonperturbative calculations since at no point in the previous discussion we *needed* to resort to a perturbative expansion: Eq. (2.3.21) can equally be viewed as the Dyson-Schwinger equation for the full self-energy, which is nonperturbative and exact.

- Alternatively, one can explicitly subtract the divergent $1/\varepsilon$ terms order by order in perturbation theory, which defines the **MS scheme** (minimal subtraction). In that case our definition of the renormalization scale μ is no longer available and $M = \mu$ takes its place instead, since it is not cancelled by the subtraction anymore.

- Another possibility is to subtract not only the divergences but *all* terms that are independent of M; this defines the $\overline{\textbf{MS}}$ **scheme** (modified minimal subtraction).

As a consequence, the masses and couplings depend not only on the renormalization point but also on the renormalization scheme. For example, the Particle Data Group (PDG) quotes the current-quark masses in the $\overline{\text{MS}}$ scheme at a renormalization scale $\mu = 2$ GeV. The quantities obtained in different schemes are related to each other by finite terms, and the invariance in the choice of μ, $m(\mu)$ and $g(\mu)$ leads to the concept of the **renormalization group**. At the end of the day, all physical observables must be independent of the renormalization point and scheme.

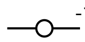
Renormalizability. So far we have only considered one explicit diagram. Do the singularities *always* cancel? Let's consider the action for a generic ϕ^p theory:

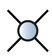
$$S = -\int d^4x \left[\frac{1}{2}\,\phi\,(\Box + m^2)\phi + \lambda_p\,\phi^p\right], \tag{2.3.59}$$

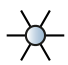
where we suppress the renormalization constants for simplicity. Now count the mass dimensions of the quantities that appear in the action:

$$[S] = 0\,, \quad [d^4x] = -4 \quad \Rightarrow \quad [\mathcal{L}] = 4\,, \quad [\phi] = 1\,, \quad [\phi^p] = p\,, \quad [\lambda_p] = 4 - p\,. \tag{2.3.60}$$

From here we can infer the dimensions of the 1PI n-point functions in momentum space:

$\Gamma_2 =$  $^{-1} = -i(p^2 - m^2) + \dots \quad \Rightarrow \quad [\Gamma_2] = 2\,,$

$\Gamma_4 =$  $\quad \Rightarrow \quad [\Gamma_4] = 0\,,$ (2.3.61)

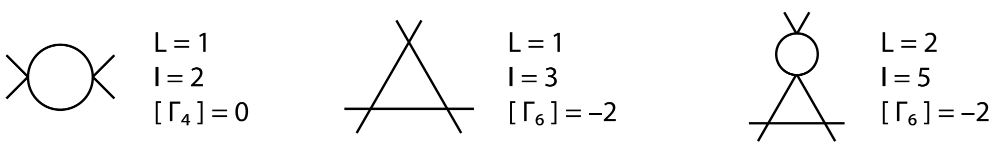
$\Gamma_6 =$  $\quad \Rightarrow \quad [\Gamma_6] = -2\,,$

because Γ_{n+2} follows from Γ_n after taking two functional derivatives $\delta^2/\delta\phi^2$. Thus, the mass dimension of Γ_n is $[\Gamma_n] = 4 - n$.

On the other hand, we can also count the dimension of Γ_n in some given order in perturbation theory. To do so, we count the number of loops L (each comes with dimension four), the number of internal propagators I (each with dimension -2), and the number of vertices (each with dimension $[\lambda_p]$):

$$[\Gamma_n] = 4L - 2I + [\lambda_p]\,V\,. \tag{2.3.62}$$

For example in ϕ^4 theory, where $[\lambda_4] = 0$:



Obviously this is consistent.

Now, the quantity $D = 4L - 2I$ also tells us how badly divergent a given diagram will be: if the number of loops L beats the number of propagators I it will diverge; if there are many propagators in a loop it will converge. D is called the **superficial degree of divergence**: if $D < 0$ the diagram converges, if $D \geq 0$ it diverges. The first diagram above has $D = 0$ and diverges logarithmically. The second has $D = -2$ and is convergent; the third has $D = -2$ but unfortunately it is still divergent because it contains a divergent subdiagram (the one on the left). Hence the name 'superficial' degree of divergence:

- a diagram with $D \geq 0$ can still be finite due to cancellations,
- a diagram with $D < 0$ can be divergent if it contains divergent subdiagrams,
- tree-level diagrams have $D = 0$ but they are finite.

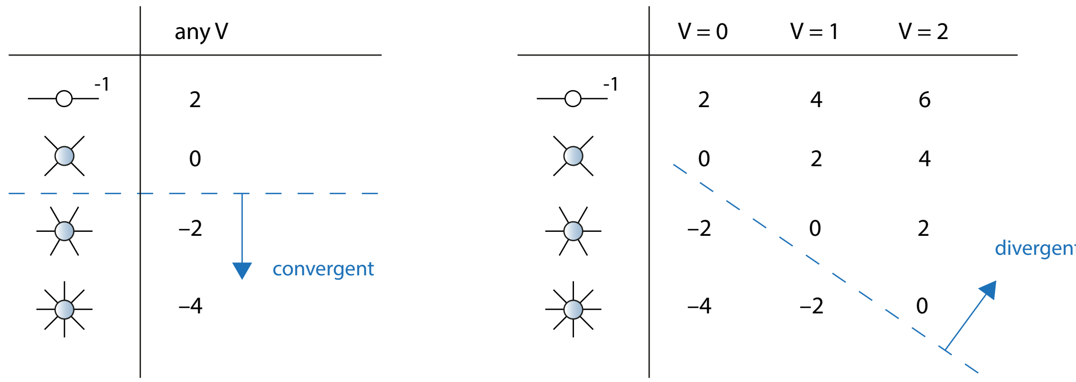


FIG. 2.12: Degree of divergence D in ϕ^4 theory (left) and ϕ^6 theory (right).

Let us ignore these subtleties for a moment and assume that D counts the actual degree of divergence. Then from Eq. (2.3.62) the degree of divergence of a given Γ_n in ϕ^p theory (with $\lambda_p = \lambda$) is

$$D = [\Gamma_n] - [\lambda]\, V\,. \tag{2.3.63}$$

The mass dimension $[\Gamma_n]$ is fixed, so depending on the mass dimension $[\lambda]$ of the coupling, D can rise or fall with higher orders in perturbation theory (expressed by V). Take ϕ^4 theory in the left panel of Fig. 2.12, where $[\lambda] = 0$ and $D = [\Gamma_n]$ is independent of V. In this case there are only two divergent n-point functions, namely the inverse propagator and the four-point function. These are also the ones with a tree-level term in the Lagrangian; they are called the **primitively divergent n-point functions**.

One can indeed show that the analysis goes through in general, also for divergent subdiagrams, which is known as the **BPHZ theorem** (Bogoliubov, Parasiuk, Hepp, Zimmermann). The reason is that the Z_i factors in front of the diagrams (which we can neglect at one-loop) cancel the divergences at higher orders. Take for example the two rightmost diagrams below Eq. (2.3.62): both contribute to the six-point function, one with $V = 3$ and the other with $V = 4$. The $V = 3$ diagram carries factors $Z = 1 + \delta Z$, where δZ contributes at higher order to the $V = 4$ graph. The sum of all contributions at a given order cancels the divergences. Here it is especially useful to employ the counterterm language, because the subdivergences will cancel with the counterterms at each order in perturbation theory.

On the other hand, the same analysis for ϕ^6 theory, where $[\lambda] = -2$ and thus $D = [\Gamma_n] + 2V$, gives the result in the right panel of Fig. 2.12: if we go high enough in perturbation theory, eventually *every* n-point function will diverge!

This leads to the notion of **renormalizability**: a theory is renormalizable if only a *finite* number of Green functions have $D \geq 0$, so that only a finite number of renormalization conditions are necessary to remove the divergences from the theory. From Eq. (2.3.63) this is equivalent to the following statement:

$$\text{A theory is renormalizable if} \quad [\lambda] \geq 0\,.$$

Thus, the coupling must either be dimensionless or have a positive mass dimension (in the latter case the theory is called *super-renormalizable*).

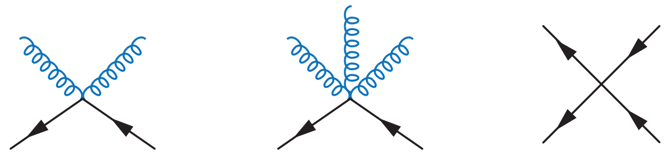


FIG. 2.13: Examples for non-renormalizable interactions constructed from fermions and gauge bosons. From left to right, the diagrams carry mass dimensions 5, 6 and 6.

A **non-renormalizable theory** has a coupling with negative mass dimension: in that case every n-point function eventually becomes divergent. Here we would need new renormalization conditions at each order in perturbation theory, and eventually infinitely many, so we must specify infinitely many constants from outside — the theory thereby loses its predictive power.

On the other hand, we will see in Sec. 4.4 that non-renormalizable theories are still perfectly acceptable **low-energy effective theories** since higher loop diagrams also come with higher momentum powers. For example, chiral perturbation theory is a non-renormalizable low-energy expansion of QCD; the non-renormalizable Fermi theory of weak interactions is the low-energy limit of the electroweak theory. In this sense, non-renormalizable theories are merely 'less fundamental' since they are not applicable at all energy scales.

Another caveat is that all considerations above are based on perturbation theory. For example, the Einstein-Hilbert action in quantum gravity defines a non-renormalizable gauge theory, which is also the reason why it is not considered as a part of the Standard Model and which has spurred developments e.g. in string theory. There is still the possibility that a non-renormalizable theory becomes *non-perturbatively* renormalizable, i.e., it 'renormalizes itself' by developing nontrivial UV fixed points. This leads to the concept of **asymptotic safety**, and there are indications that this is what could be at play in quantum gravity.

In any case, a renormalizable QFT contains only a small number of superficially divergent amplitudes, namely those with a tree-level counterpart in the Lagrangian, and therefore it only needs a finite number of renormalization constants. The good news is that we can read off a theory's renormalizability directly from its Lagrangian: we just need to look at the mass dimension of the coupling constant. For a scalar ϕ^p theory only ϕ^3 and ϕ^4 interactions are renormalizable whereas those with $p > 4$ are not. Likewise, the QCD Lagrangian is renormalizable, whereas diagrams such as in Fig. 2.13 are not: with $[\psi] = 3/2$ and $[A] = 1$, their mass dimensions are greater than 4, and to compensate this we would need to attach couplings with negative mass dimensions. Renormalizability restricts the possible forms of interactions dramatically!

2.3.3 β function and running coupling

Callan-Symanzik equation. Consider again a generic field theory with a field ϕ or several fields ϕ_i. Then the bare and renormalized fields are related by $\phi_B = Z_\phi^{1/2}\,\phi$ in analogy to Eq. (2.3.1). Since this implies

$$\frac{\delta^n\Gamma}{\delta\varphi_B^n} = Z_\phi^{-n/2}\,\frac{\delta^n\Gamma}{\delta\varphi^n}\,, \tag{2.3.64}$$

we can read off how a renormalized 1PI correlation function ($\Gamma^n = \delta^n\Gamma/\delta\varphi^n$), which depends on a set of momenta $\{p_i\}$, the renormalized coupling g, the renormalized mass m and the renormalization point μ, is related to its bare counterpart ($\Gamma^n_{\mathrm{B}} = \delta^n\Gamma/\delta\varphi^n_{\mathrm{B}}$):

$$\Gamma^n(\{p_i\}, g, m, \mu) = Z_\phi^{n/2}\,\Gamma^n_{\mathrm{B}}(\{p_i\}, g_{\mathrm{B}}, m_{\mathrm{B}})\,. \tag{2.3.65}$$

The bare quantities cannot depend on the renormalization scale μ. If we apply the derivative $\mu\,d/d\mu$ and use $d\Gamma^n_{\mathrm{B}}/d\mu = 0$, we obtain the **Callan-Symanzik equation**:

$$\left(\mu\,\frac{\partial}{\partial\mu} + \underbrace{\mu\,\frac{dg}{d\mu}}_{\beta(g)}\,\frac{\partial}{\partial g} + \underbrace{\mu\,\frac{dm}{d\mu}}_{m\gamma_m(g)}\,\frac{\partial}{\partial m}\right)\Gamma^n = \mu\,\frac{n}{2}\,Z_\phi^{n/2-1}\,\frac{dZ_\phi}{d\mu}\,\Gamma^n_{\mathrm{B}} = n\,\underbrace{\frac{\mu}{2}\,\frac{d\ln Z_\phi}{d\mu}}_{\gamma(g)}\,\Gamma^n\,. \tag{2.3.66}$$

Here we defined the β function $\beta(g)$, the anomalous mass dimension $\gamma_m(g)$, and the anomalous dimension of the field $\gamma(g)$; they determine the respective change of the coupling, the mass and the field renormalization under a change of the renormalization scale. For n-point functions that depend on more than one field we would have to include a separate $\gamma(g)$ for each of them.

The Callan-Symanzik equation entails that a shift of the renormalization scale can be compensated by an appropriate shift of the coupling, the mass and the fields. Suppose for the moment that $\gamma(g) = 0$, so that Z_ϕ is independent of μ. We also set $m = 0$ to simplify the discussion. The l.h.s of the equation then implies $d\Gamma^n/d\mu = 0$, i.e. also the renormalized n-point function is $\mu-$independent. A change of the renormalization point can then always be compensated by a shift of the coupling:

$$\Gamma^n(\{p_i\}, g(\mu), \mu) = \Gamma^n(\{p_i\}, g(\mu_0), \mu_0)\,. \tag{2.3.67}$$

Moreover, the Callan-Symanzik equation also allows us to compensate the *momentum* dependence of an n-point function by a change in its coupling. Consider an n-point function with mass dimension D; it can be written as

$$\Gamma^n(\{p_i\}, g(\mu), \mu) = \mu^D f\left(\left\{\frac{p_i}{\mu}\right\},\, g(\mu)\right) = \mu_0^D\, f\left(\left\{\frac{p_i}{\mu_0}\right\},\, g(\mu_0)\right), \tag{2.3.68}$$

where the function f is dimensionless. The first equality is simply a dimensional argument, and the second follows from Eq. (2.3.67) since the expression is independent of μ. Now replace all momenta $p_i \to \lambda p_i$, where $\lambda = \mu/\mu_0$:

$$f\left(\lambda\left\{\frac{p_i}{\mu_0}\right\},\, g(\mu_0)\right) = \lambda^D f\left(\left\{\frac{p_i}{\mu_0}\right\},\, g(\lambda\mu_0)\right). \tag{2.3.69}$$

Hence, at a fixed renormalization point μ_0, a uniform rescaling of momenta can be compensated by an according shift of the coupling on which the Green function depends. If we dropped our simplifications $\gamma(g) = 0$ and $m = 0$, the equation would pick up a scaling factor that depends on $\gamma(g)$, and the renormalized mass would obtain a scaling factor $\sim \gamma_m(g)$, hence the name 'anomalous dimensions'.

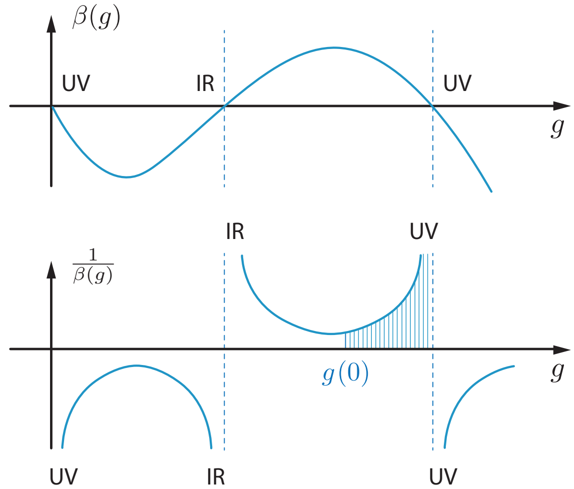


FIG. 2.14: Possible shape of the β function and its inverse that appears in Eq. (2.3.70).

β function. The β function of a theory, $\beta(g) = \mu\, dg/d\mu$, encodes the change of the running coupling with the momentum scale. If we change the scale from μ_0 to μ and define the dimensionless variable $t = \ln(\mu/\mu_0) \in [-\infty, \infty]$, which entails $\mu\, d/d\mu = d/dt$, then the change from the coupling $g(0)$ at μ_0 to $g(t)$ at μ is given by

$$\beta(g) = \frac{dg(t)}{dt} \quad \Rightarrow \quad \int\limits_{g(0)}^{g(t)} \frac{dg}{\beta(g)} = \int\limits_0^t dt' = t\,, \tag{2.3.70}$$

which can be solved for $g(t)$ if $\beta(g)$ is known.

To understand this equation better, let us study possible shapes of the β function (Fig. 2.14). The values $g^\star$ where $\beta(g^\star) = 0$ are **fixed points** under a renormalization-group evolution because the coupling does not change in the vicinity of $g^\star$ ($dg/dt = 0$). Eq. (2.3.70) entails that the l.h.s. must diverge for $t \to \pm\infty$: this happens when $g(t)$ runs into the fixed point nearest to $g(0)$, or when it goes to infinity because there is no zero of $\beta(g)$ to approach. Whether the fixed point corresponds to $t \to \infty$ or $t \to -\infty$ depends on the sign of the β function and the integration direction:

- An **ultraviolet (UV) fixed point** ($t \to +\infty$) implies $g(t) > g(0)$ and $\beta > 0$ or $g(t) < g(0)$ and $\beta < 0$;
- An **infrared (IR) fixed point** ($t \to -\infty$) implies $g(t) > g(0)$ and $\beta < 0$ or $g(t) < g(0)$ and $\beta > 0$.

The origin $g = 0$ is always a fixed point since $\beta(0) = 0$. A theory is called

- **asymptotically free** if $g = 0$ is a UV fixed point, because then the coupling becomes small for $t \to \infty$ (as we will see below, this is the case for QCD);
- **infrared stable** if $g = 0$ is an IR fixed point (e.g. QED, ϕ^4 theory).

The domains separated by fixed points correspond to different theories, unless there are several couplings in the theory (in which case one ends up with a multidimensional phase diagram).

Calculation of the β function. In the following we sketch the calculation of the β function in QCD (for which Gross, Politzer and Wilczek received the Nobel Prize in 2004). We start with the relation $g_B = Z_g\, g$ from Eq. (2.3.1), where the bare coupling g_B does not depend on μ. In four dimensions g is dimensionless, but since we want to employ dimensional regularization we must work out the dimension $[g]$ of the coupling in $d = 4-\varepsilon$ dimensions. Because the action remains dimensionless and the spacetime integral is $d^d x$, we have

$$[\mathcal{L}] = d\,, \quad [\psi] = \frac{d-1}{2}\,, \quad [A] = \frac{d-2}{2} \quad \Rightarrow \quad [g] = [\mathcal{L}] - [\bar\psi \not{A}\, \psi] = \frac{\varepsilon}{2}\,. \tag{2.3.71}$$

Thus we write $g_B = Z_g\, g\, \mu^{\varepsilon/2}$, where g is the dimensionless coupling in arbitrary dimensions (this is equivalent to putting factors μ^ε in front of loop integrals such as Eq. (2.3.48)). The β function then becomes

$$\begin{aligned}\beta(g) = \frac{dg}{dt} &= \mu\,\frac{d}{d\mu}\left(\frac{g_\text{B}}{Z_g\,\mu^{\varepsilon/2}}\right) = \mu\left(-\frac{\varepsilon}{2}\,\frac{g_B}{Z_g\,\mu^{\varepsilon/2+1}} - \frac{1}{Z_g^2}\,\frac{dZ_g}{d\mu}\,\frac{g_B}{\mu^{\varepsilon/2}}\right) \\ &= -\left(\frac{\varepsilon}{2} + \frac{d}{dt}\ln Z_g\right) g\,.\end{aligned} \tag{2.3.72}$$

To proceed, we must calculate the g dependence of Z_g. From Eq. (2.3.3) we see that Z_g appears in all vertices in the Lagrangian in combination with other renormalization constants, so we could obtain it from any of the combinations $\{Z_A, Z_\psi, Z_\Gamma\}$, $\{Z_A, Z_c, \widetilde{Z}_\Gamma\}$, $\{Z_A, Z_{3g}\}$ or $\{Z_A, Z_{4g}\}$. In the first case, we must calculate the one-loop diagrams for the gluon propagator, the quark propagator and the quark-gluon vertex. Because the renormalization constants absorb the infinities, the simplest option is to use the MS scheme where they only absorb the $1/\varepsilon$ terms and nothing else. For example, for the quark propagator we have from Eqs. (2.3.21) and (2.3.52):

$$A(p^2) = Z_\psi + \Sigma_A(p^2) = Z_\psi + \frac{\alpha}{2\pi}\,C_F \int dx\, x\left(\frac{2}{\varepsilon} - \gamma + \ln\frac{4\pi M^2}{\Delta} - 1\right). \tag{2.3.73}$$

In our earlier MOM scheme we demanded $A(\mu^2) = 1$, which led to $Z_\psi = 1 - \Sigma_A(\mu^2)$, whereas in the MS scheme we only subtract the infinities:

$$Z_\psi = 1 - \frac{\alpha}{2\pi}\,C_F \int dx\, x\,\frac{2}{\varepsilon} = 1 - \frac{g^2}{(4\pi)^2}\,\frac{2C_F}{\varepsilon}\,. \tag{2.3.74}$$

In the same way one computes Z_A and Z_Γ, where only the highest momentum powers in the loop integrals contribute since only those produce the divergences and thus the $1/\varepsilon$ terms. Putting everything together, the one-loop result for Z_g becomes

$$Z_g = 1 - \frac{b}{\varepsilon}\,g^2 + \dots\,, \qquad b = \frac{\beta_0}{(4\pi)^2}\,, \qquad \beta_0 = 11 - \frac{2}{3}\,N_f\,, \tag{2.3.75}$$

where N_f is the number of flavors. Inserting this back into Eq. (2.3.72) gives

$$\begin{aligned}\frac{d}{dt}\,\ln\,Z_g = -\frac{2b}{\varepsilon}\,g\,\beta(g) + \dots \;\Rightarrow\quad &\beta(g) = -\left(\frac{\varepsilon}{2} - \frac{2b}{\varepsilon}\,g\,\beta(g)\right) g \\ \Rightarrow\quad &\beta(g)\left(1 - \frac{2b}{\varepsilon}\,g^2\right) = -\frac{\varepsilon g}{2} \\ \Rightarrow\quad &\beta(g) = -\frac{\varepsilon g}{2} - bg^3 + \dots\end{aligned} \tag{2.3.76}$$

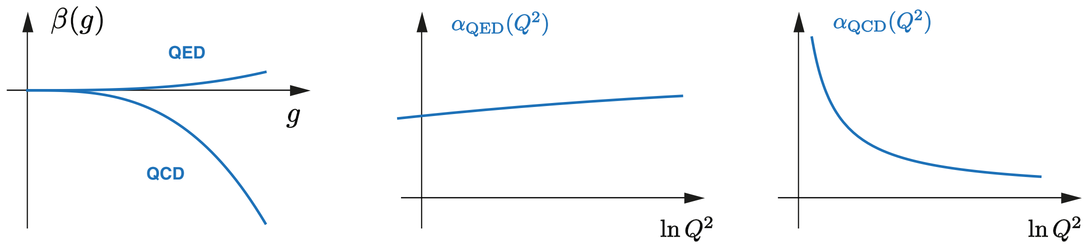


FIG. 2.15: β function in QCD and QED (left) and resulting shapes of the running coupling.

For $\varepsilon \to 0$, we obtain the result

$$\beta(g) = -bg^3 + \mathcal{O}(g^5)\,. \tag{2.3.77}$$

The negative sign of the β function at $g \to 0$ shows that QCD is indeed an asymptotically free theory, i.e., $g(t)$ becomes small at large momenta $t \to \infty$. Note that this is only true for $\beta_0 > 0$, which entails $N_f \leq 16$; for more than 16 flavors we would lose asymptotic freedom. The lowest-order coefficients at $\mathcal{O}(g^3)$ and $\mathcal{O}(g^5)$ are independent of the renormalization scheme, whereas higher-order terms are not.

Running coupling. If we put the result for $\beta(g)$ back into Eq. (2.3.70), we obtain the running coupling of QCD:

$$\int\limits_{g(0)}^{g(t)} \frac{dg}{-bg^3} = \frac{1}{2b}\left(\frac{1}{g(t)^2} - \frac{1}{g(0)^2}\right) = t \quad \Rightarrow \quad g(t)^2 = \frac{g(0)^2}{1+2bt\,g(0)^2}\,, \tag{2.3.78}$$

or equivalently $\alpha(t) = g(t)^2/(4\pi) = \alpha(0)/[1+\frac{\beta_0}{4\pi}\,\alpha(0)\,2t]$. Writing $2t = \ln(\mu^2/\mu_0^2)$, this expression has a pole at $\mu^2 = \Lambda^2_{\rm QCD}$ defined by

$$\alpha(0) = \frac{1}{\frac{\beta_0}{4\pi}\,\ln\frac{\mu_0^2}{\Lambda^2_{QCD}}} \quad \Rightarrow \quad \alpha(t) = \frac{1}{\frac{\beta_0}{4\pi}}\,\frac{1}{\ln\frac{\mu_0^2}{\Lambda^2_{\rm QCD}} + \ln\frac{\mu^2}{\mu_0^2}} = \frac{1}{\frac{\beta_0}{4\pi}\,\ln\frac{\mu^2}{\Lambda^2_{\rm QCD}}}\,. \tag{2.3.79}$$

From the Callan-Symanzik equation we can interpret the dependence on μ^2 as a dependence on q^2. Actually we should have started from large *spacelike* ('Euclidean') momenta $q^2 = -Q^2 < 0$, because this is the momentum region where we can compare to experiment and where $\alpha(Q^2)$ is guaranteed to be free of singularities. As long as μ^2 and μ_0^2 are also spacelike, this does not change the formulas and we arrive at

$$\alpha(Q^2) = \frac{1}{\frac{\beta_0}{4\pi}\,\ln(Q^2/\Lambda^2_{\rm QCD})}\,. \tag{2.3.80}$$

At large momenta where $\alpha(Q^2)$ becomes small, quarks and gluons behave as asymptotically free particles and we can apply perturbation theory. On the other hand, this also means that the coupling increases at small momenta and perturbation theory will eventually fail. In that region, nonperturbative effects related to the formation of bound states become important.

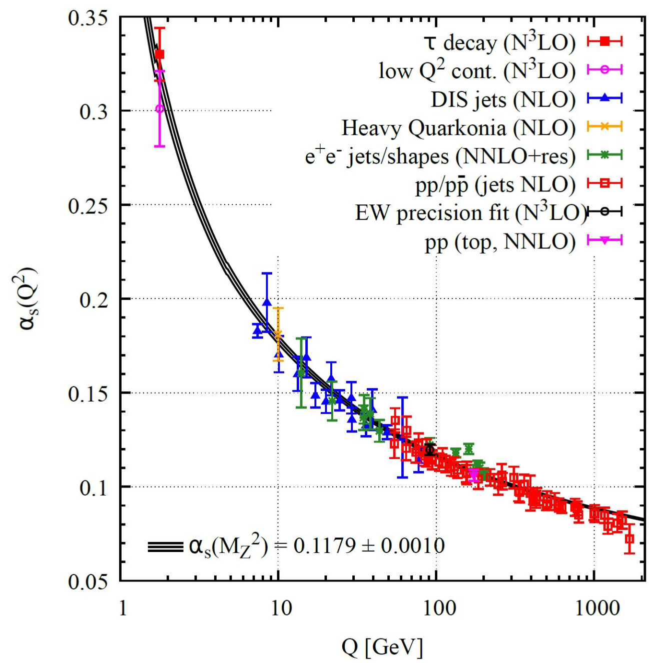


Fig. 2.16: Overview of $\alpha(Q^2)$ measurements from the PDG, taken from P. A. Zyla et al., Prog. Theor. Exp. Phys. **2020**, 083C01 (2020).

The analogous calculation in QED gives $\beta_0 = -4/3$ so that $\beta(g \to 0)$ is positive: QED is infrared stable and the coupling grows with increasing momenta. It actually grows very slowly (Fig. 2.15), so that perturbation theory works very well over many orders of magnitude.

Λ_{QCD} is the scale where perturbation theory definitely breaks down since it produces an unphysical Landau pole in the perturbative expansion. Eq. (2.3.80) and its refinements at higher loop orders allow one to convert the running coupling at a given scale — see Fig. 2.16 for the current world average of $\alpha(M_Z^2)$ — to a value for Λ_{QCD}, which therefore depends on the order in perturbation theory, the renormalization scheme, and the number of active flavors at the scale where the coupling is probed (due to the N_f dependence in β_0). Comparison of $\alpha(Q^2)$ at four-loop order with experimental results yields the value $\Lambda_{\overline{\text{MS}}}^{N_f=5} = 210(14)$ MeV [PDG 2018].

Alternative calculation of the running coupling. Another way to compute the running coupling is to start from the *finite* quantities (i.e., the renormalized propagators and vertices) instead of the divergent ones (the renormalization constants). To do so, note that the renormalization constants do not only relate the renormalized with the bare fields, but also the renormalized dressing functions of the propagators and vertices with their bare counterparts, cf. Eq. (2.3.65). For the gluon and ghost propagator and the ghost-gluon vertex this reads:

$$Z_B(q^2) = Z_A\, Z(q^2)\,, \quad G_B(q^2) = Z_c\, G(q^2)\,, \quad \Gamma_{\text{gh}}(q^2) = \widetilde{Z}_\Gamma\, \Gamma^B_{\text{gh}}(q^2)\,. \tag{2.3.81}$$

Here, $\Gamma_{\text{gh}}(q^2) = \widetilde{f}_1(q^2, q^2, q^2)$ is the dressing function attached to the tree-level tensor of the ghost-gluon vertex in Eq. (2.3.17). We also have $g_B = Z_g\, g$ and thus $\alpha_B = Z_g^2\, \alpha$.

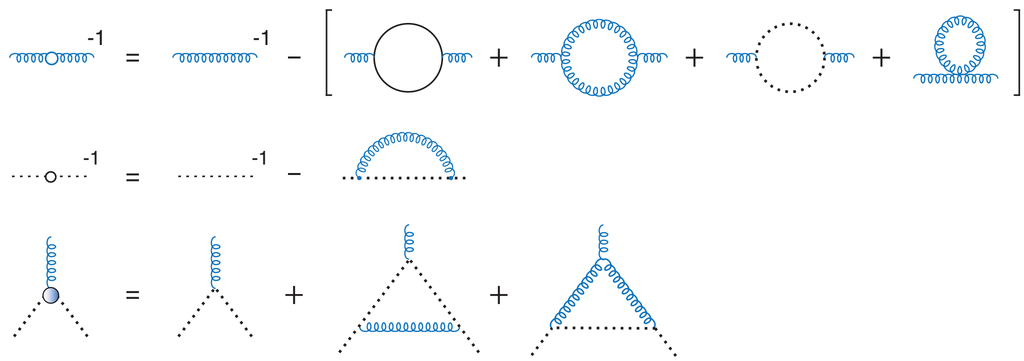


FIG. 2.17: One-loop diagrams for the gluon and ghost propagator and the ghost-gluon vertex.

We can use Eq. (2.3.3) to find combinations that stay unrenormalized, i.e. for which $F_B = F$, since only those can contain observable information. One such combination is the 'running coupling from the ghost-gluon vertex':

$$\bar{\alpha}(q^2) = \alpha\, Z(q^2)\, G^2(q^2)\, \Gamma_{\text{gh}}^2(q^2) = \underbrace{\frac{\widetilde{Z}_\Gamma^2}{Z_g^2\, Z_A\, Z_c^2}}_{=1}\, \bar{\alpha}_B(q^2)\,. \tag{2.3.82}$$

The bare quantities are individually divergent but the divergences must cancel in the combination.

To determine $\bar{\alpha}(q^2)$, one must calculate the one-loop diagrams in Fig. 2.17 for the gluon propagator, the ghost propagator and the ghost-gluon vertex. The results for a general gauge parameter ξ are

$$\begin{aligned} Z(q^2) &= 1 - \frac{\alpha}{4\pi}\left[\frac{N_c}{2}\left(\frac{13}{3}-\xi\right) - \frac{4}{3}\,T_F N_f\right] \ln\frac{q^2}{\mu^2}\,, \\ G(q^2) &= 1 - \frac{\alpha}{4\pi}\left[N_c\,\frac{3-\xi}{4}\right] \ln\frac{q^2}{\mu^2}\,, \\ \Gamma_{\text{gh}}(q^2) &= 1 - \frac{\alpha}{4\pi}\left[N_c\,\frac{\xi}{2}\right] \ln\frac{q^2}{\mu^2}\,. \end{aligned} \tag{2.3.83}$$

The first term in the bracket for $Z(q^2)$ is the sum of the gluon and ghost loop (the tadpole drops out). The second term comes from the quark loop, where the color trace is $T_F = 1/2$ in the fundamental representation and we set $q^2, \mu^2 \gg m^2$. Taking the squares of $G(q^2)$ and $\Gamma_{\text{gh}}(q^2)$, the terms in the brackets add up to

$$\beta_0 = \frac{N_c}{2}\left(\frac{13}{3}-\xi\right) - \frac{4}{3}\,T_F N_f + N_c\,\frac{3-\xi}{2} + N_c\,\xi = \frac{11}{3}\,N_c - \frac{4}{3}\,T_F N_f\,, \tag{2.3.84}$$

where the dependence on the gauge parameter ξ has dropped out. This is identical to the result (2.3.75) and the resulting running coupling at one-loop order is

$$\bar{\alpha}(q^2) = \alpha\left(1 - \frac{\alpha}{4\pi}\,\beta_0\,\ln\frac{q^2}{\mu^2} + \dots\right) \approx \frac{\alpha}{1 + \alpha\,\frac{\beta_0}{4\pi}\,\ln\frac{q^2}{\mu^2}}\,. \tag{2.3.85}$$

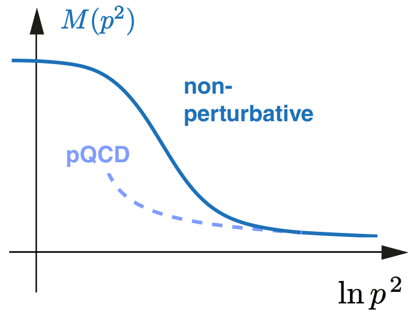


FIG. 2.18: Qualitative shape of the quark mass function from perturbation theory and non-perturbative calculations.

In QED, in the absence of gauge-boson self-interactions and ghosts, the relations in Eq. (2.3.3) reduce to $Z_g Z_A^{1/2} = 1$ and $Z_\Gamma = Z_\psi$, so the analogous definition of the running coupling is $\bar{\alpha}(q^2) = \alpha\, Z(q^2)$. In that case only the diagram with the fermion loop in the photon propagator survives, which yields $\beta_0 = -4/3$ for one species of fermions. In Eq. (2.3.84) one can see how the screening effect from the quark loop, which gives a negative contribution to β_0 for $N_f \leq 16$, is overwhelmed by the 'antiscreening' from the remaining diagrams involving gluons and ghosts.

Running quark mass. The **quark mass function** is another combination that stays unrenormalized since Eq. (2.3.11) entails

$$iS^{-1}(p) = A(p^2)\left(\not{p} - M(p^2)\right) = Z_\psi\, iS_B^{-1}(p) = Z_\psi\, A_B(p^2)\left(\not{p} - M_B(p^2)\right) \tag{2.3.86}$$

and thus $M(p^2) = M_B(p^2)$. We already worked out the one-loop result for the mass function in Eq. (2.3.58). If we define the anomalous mass dimension γ_m as

$$\gamma_m = \frac{3C_F}{\beta_0} = \frac{4}{11 - \frac{2}{3}N_f}\,, \tag{2.3.87}$$

then we can write to one-loop order for $\alpha \ll 1$:

$$\begin{aligned} M(p^2) &= m\left[1 - \frac{3\alpha}{4\pi}\,C_F\,\ln\frac{p^2}{\mu^2}\right] = m\left[1 - \gamma_m\,\frac{\beta_0}{4\pi}\,\alpha\,\ln\frac{p^2}{\mu^2}\right] \\ &= m\left[1 + \frac{\beta_0}{4\pi}\,\alpha\,\ln\frac{p^2}{\mu^2}\right]^{-\gamma_m} = m\left[\frac{\alpha(p^2)}{\alpha(\mu^2)}\right]^{\gamma_m} \\ &= m\left[\frac{\frac{1}{2}\ln(\mu^2/\Lambda^2_{\text{QCD}})}{\frac{1}{2}\ln(p^2/\Lambda^2_{\text{QCD}})}\right]^{\gamma_m} = \frac{\hat{m}}{\left[\frac{1}{2}\ln(p^2/\Lambda^2_{\text{QCD}})\right]^{\gamma_m}}\,. \end{aligned} \tag{2.3.88}$$

This gives the one-loop running of the quark mass function at large p^2. It is also independent of the gauge parameter ξ, whereas the result for $A(p^2)$ is

$$A(p^2) = 1 - \xi\,C_F\,\frac{\alpha}{4\pi}\,\ln\frac{p^2}{\mu^2}\,. \tag{2.3.89}$$

Unfortunately, QCD perturbation theory turns out to be of limited use in this case because for light quarks the biggest contribution to the mass function $M(p^2)$ is generated **non-perturbatively** by spontaneous chiral symmetry breaking (Fig. 2.18). We will return to this point in Sec. 4.2.

From Eq. (2.3.83) we can also read off the anomalous dimensions for the gluon and ghost propagators and the ghost-gluon vertex. Writing

$$Z(q^2) \propto \frac{1}{[\ln(q^2/\Lambda^2)]^{\gamma_{\text{gl}}}}\,, \quad G(q^2) \propto \frac{1}{[\ln(q^2/\Lambda^2)]^{\gamma_{\text{gh}}}}\,, \quad \Gamma_{\text{gh}}(q^2) \propto \frac{1}{[\ln(q^2/\Lambda^2)]^{\gamma_{\text{gh-gl}}}} \tag{2.3.90}$$

we find

$$\gamma_{\text{gl}} = \frac{1}{\beta_0}\left[\frac{N_c}{2}\left(\frac{13}{3}-\xi\right)-\frac{4}{3}\,T_F N_f\right], \quad \gamma_{\text{gl}} = \frac{1}{\beta_0}\left[N_c\,\frac{3-\xi}{4}\right], \quad \gamma_{\text{gh-gl}} = \frac{1}{\beta_0}\left[N_c\,\frac{\xi}{2}\right], \tag{2.3.91}$$

where $\gamma_{\text{gl}} + 2\gamma_{\text{gh}} + 2\gamma_{\text{gh-gl}} = 1$. In Landau-gauge Yang-Mills theory ($\xi = 0$, $N_f = 0$) this reduces to

$$\gamma_{\text{gl}} = \frac{13}{22}\,, \qquad \gamma_{\text{gh}} = \frac{9}{44}\,, \qquad \gamma_{\text{gh-gl}} = 0\,. \tag{2.3.92}$$

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